U3A - Maths

Term 2 / week 4 — 14/05/21

Pythagorean Means

We have,

Arithmetic Mean (A.M.) $= \left(\dfrac{x_1 + x_2 + \dots + x_n}{n}\right)$

Geometric Mean (G.M.) $= \sqrt[n]{x_1 \, x_2 \dots x_n}$

Harmonic Mean (H.M.) $= \dfrac{n}{(1/x_1 + 1/x_2 + \dots + 1/x_n)}$

- Pythagoras (570 - 495 BC) not only used the above means, but he also established a relationship between them using geometry!

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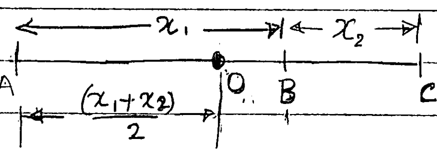
Let us consider $n = 2$,

A.M. $= \dfrac{x_1 + x_2}{2}$; G.M. $= \sqrt{x_1 x_2}$; H.M. $= \dfrac{2 x_1 x_2}{(x_1 + x_2)}$

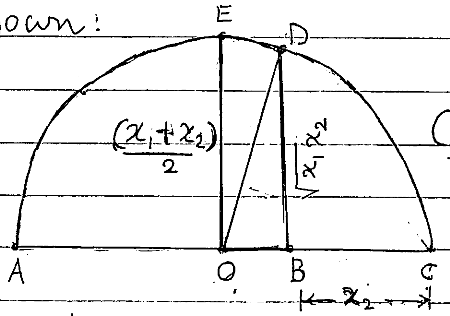
Expressing graphically



(Fig 1)

AM is the midpoint 'O' of line 'AC'

Draw a semicircle with centre at x_A as shown:



(Fig. 2)

We have, Arithmetic Mean = "Radius"

Geometric Mean = distance "BD" (Vertical at B)

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Let us now prove vertical distance 'BD' is the Geom. Mean.

- Let us first draw the line 'OD'

We have,

$$'OD' = \text{Radius} = \frac{(x_1 + x_2)}{2}$$

(See Fig 1) $'OB' = 'AB' - 'AO'$

$$= x_1 - \frac{(x_1 + x_2)}{2}$$

Using (the famous)! Pythagorean theorem

$$(BD)^2 = (OD)^2 - (OB)^2$$

$$= \left(\frac{x_1 + x_2}{2}\right)^2 - \left[x_1 - \left(\frac{x_1 + x_2}{2}\right)\right]^2$$

$$= \cancel{\left(\frac{x_1 + x_2}{2}\right)^2} - \left[x_1^2 + \cancel{\left(\frac{x_1 + x_2}{2}\right)^2} - \frac{\cancel{2} \cdot x_1 (x_1 + x_2)}{\cancel{2}}\right]$$

$$= 0 - [\cancel{x_1^2} - \cancel{x_1^2} - x_1 x_2]$$

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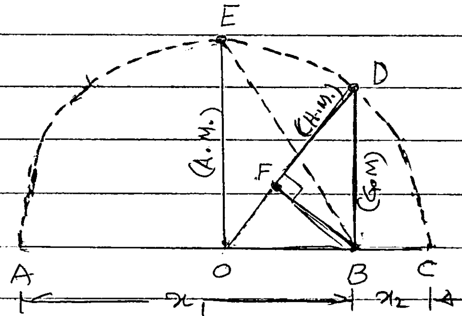
$$= + x_1 x_2$$

$\therefore$ Geometric Mean $= 'BD' = \sqrt{x_1 x_2}$!!

- To find Harmonic mean (H.M.)



(Fig 4)

- Draw the perpendicular BF on radius line OD.

$$\text{Harmonic Mean} = DF = \frac{2}{(1/x_1 + 1/x_2)} = \frac{2 x_1 x_2}{(x_1 + x_2)}$$

Note: The ratio $x_1 : x_2$ in Fig 4 has been changed for clarity. Proof of H.M. is homework!!

Inspecting Fig. 4, we have

'OE' > 'BD' > 'DF'

or

Arith Mean > Geom Mean > Harm. Mean!

Finally, referring to Fig 4, we have

Quadratic Mean or Root Mean Square Value

= length of line "BE"

we have,

$$'BE' = \sqrt{\left(\frac{x_1+x_2}{2}\right)^2 + \left(\frac{x_1-x_2}{2}\right)^2}$$

$$= \sqrt{\frac{x_1^2+x_2^2}{2}} = \sqrt{\frac{\sum x_i^2}{n}}$$ R.M.S. Value!

Relevance of Arithmetic & Geometric Mean

Ex. 1. Calculate A.M., Median & G.M. for the following data.

1, 4, 7, 10, 13, 16, 19

$A.M. = (1+4+7+10+13+16+19)/7$

$= 10$

Median = "Middle" value = 10

$G.M. = \sqrt[7]{1 \times 4 \times 7 \times 10 \times 13 \times 16 \times 19}$

$= \sqrt[7]{1,106,560} = 7.3$

Note: The A.M. is closer to the "middle" value compared with G.M. — since the data is linear!

Ex. 2 Calculate A.M., Median & G.M. for the following data.

1, 3, 9, 27, 81, 243, 729

$A.M. = (1+3+9+27+81+243+729)/7$

$= 156.14$

Median = "Middle" Value = 27

$G.M. = \sqrt[7]{1 \times 3 \times 9 \times 27 \times 81 \times 243 \times 729}$

$= \sqrt[7]{1.046 \times 10^{10}} = 27$

Note: The G.M. is closer to the middle value compared with A.M. — since the values are "multiplicative" or "exponential"

Geometric Mean is also useful for calculating the "average" of unrelated values

Ex. 3 We wish to compare

Coffee Shop "A" : Source 1 rating 4.5/5
Source 2 rating 68/100

Coffee Shop "B" : Source 1 rating 3/5
Source 2 rating 75/100

Arithmetic Mean gives

Coffee Shop 'A' : $(4.5+68)/2 = 36.25$

Coffee Shop 'B' : $(3+75)/2 = 39.00$

∴ Coffee shop 'B' is better!

Geometric Mean gives

Coffee Shop 'A' : $\sqrt{4.5 \times 68} = 17.5$

Coffee Shop 'B' : $\sqrt{3 \times 75} = 15$

Coffee shop 'A' is better!

Home Work Verify that G.M. gives the correct result by "normalising" the data!