U3A Maths

29-Jul-2022 Term 3 / Week 2

Buffon's Needle

Buffon's needle is one of the oldest problems in "geometric probability". It was first stated by Georges Louis Leclerc, in 1770.
a French mathematician, cosmologist & philosopher.
He was also the "Count of Buffon" (Compte de Buffon)

Problem Definition

If a needle or a group of needles are dropped on to a series of parallel lines (traditionally dropped on parallel wooden floor boards), then the probability that a needle lie across the line is given by:

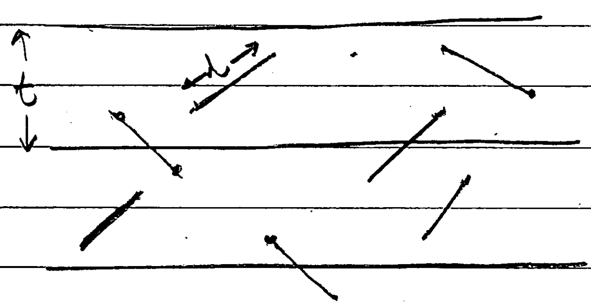
$$p = \frac{2l}{t\pi} \quad \text{for } l \le t$$

($l > t$ results in a more complex eqn)

where:

t - distance bet. parallel lines
l - length of the needle.

It is graphically illustrated as below:

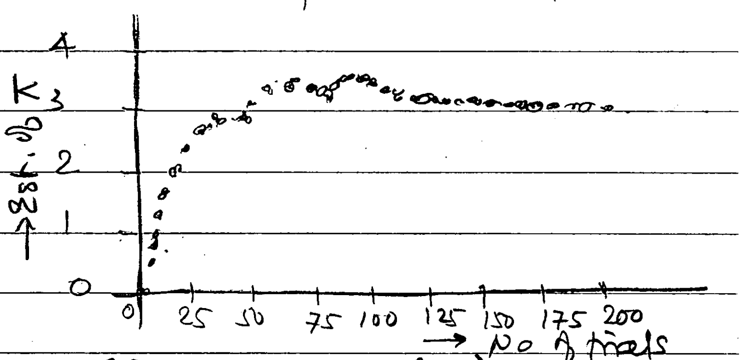


The above problem can then be used to estimate the value of π, by conducting a series of trials and establishing the probability (P)

Hence, $\pi \simeq \frac{2l}{t.p}$

Obviously, to get a good estimate, one has to conduct a large number of trials.

A computer simulation of repeated experiments with $l = 2.6$ & $t = 5.0$ resulted in the following estimate for π.



(Ref: Wikipedia)

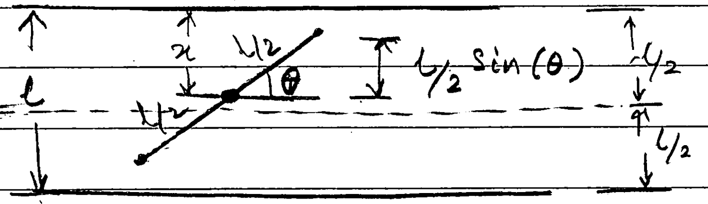
Proof

Let us do the proof in two steps. (It is generally assumed that $l < t$)

Step 1 :

Let the length of the parallel lines be the same as that of the needle.
i.e, both of them have length 'l'.

Let say the dropped needle has its mid-point

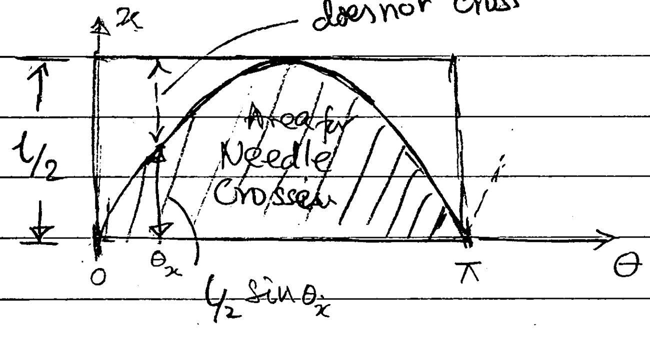


at a distance 'x', as shown in the above figure.
As per the figure shown, the needle will intersect the (top) line, provided

$$x \leqslant \tfrac{l}{2} \sin\theta$$

We will analyse the case when the needle touches/intersects the top line.

When θ varies from 0 to π, the intersection with the top line occurs when $x \leq \frac{l}{2}\sin\theta$.
It is shown graphically as below:



∴ Probability of needle crossing

$$p = \frac{\int_0^{\pi} \frac{l}{2}\sin\theta \, d\theta}{\frac{l}{2} \cdot \pi}$$

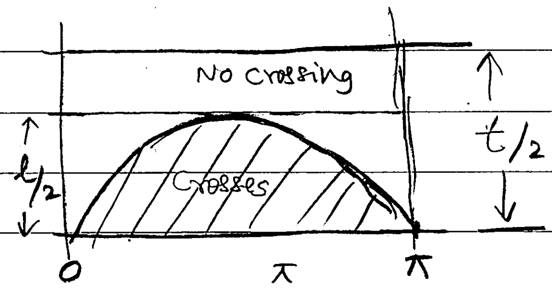
After solving, we get

$$p = \left(\frac{2}{\pi}\right)$$

Step 2

Let us now consider that parallel line distance (t) is greater than needle length (l),

The area for intersection can be shown as



$$p = \frac{\int_0^{\pi} \frac{l}{2}\sin\theta \, d\theta}{(t/2)\,\pi}$$

$$= \frac{2l}{t\pi}$$