U3A Maths

17-Feb-2023 Term 1/Week3

Imaginary Number - History

Review

- Evolution of numbers

$\{\mathbb{N}\} = \{1, 2, 3, 4, \ldots\}$

Set of Natural Numbers

(Not closed for 0 & -ve nos.)

$\{\mathbb{Z}\} = \{\{\mathbb{N}\}, 0, -1, -2, \ldots\}$

Set of Integers

(Not closed for division)

$\{\mathbb{Q}\} = \{\{\mathbb{Z}\}, \frac{1}{2}, \frac{1}{4}, \frac{2}{3}, -\frac{1}{5}, \ldots\}$

set of Rational numbers.

$\{\mathbb{I}\}$ = set of Irrational ($\sqrt{2}, \sqrt{3}/2, -\sqrt{5}, \ldots$) & Transcendental ($e, \pi, \ldots$) numbers

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The set of Real Numbers is

$\{\mathbb{R}\} = \{\{\mathbb{Q}\}, \{\mathbb{I}\}\}$

ie, set of Rational, Irrational & Transcendental Numbers.

- The set of "Real Numbers" is not closed for "square root of negative numbers. for Ex, $\sqrt{-5}$, $\sqrt{-1}$, etc
- In general, "square root of a negative number" can be written as; for Ex.

$\sqrt{-4} = \sqrt{(-1)(4)} = \sqrt{-1}\cdot\sqrt{4} = i2$ or $2i$

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Where $i = \sqrt{-1}$ - Imaginary Number?!!

- We can now define the "set of Complex Numbers" as

$\{\mathbb{C}\} = \{\{\mathbb{R}\}, i\}$

- In general, a complex number is written as,

$(a + ib)$ or $(a+bi)$

Real Part Imaginary Part

Ex: $(2+i3)$, $(1+i0)$, $(0+i2)$ etc.

- Set of Complex numbers is closed for all arithmetic operations $\Rightarrow +, -, \times, \div, \sqrt[n]{x}$ & x^n!

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- Hence, in reality, the complex number is the most general form of numbers!

=x=

- Let us now explore the origin of the need for "imaginary" no.
- In the early days, algebraic equations, such as quadratic equations, were solved using geometrical methods.
- Hence, there was no need for negative numbers, let alone complex or imaginary numbers.

Ex1. Solve: $x^2 + 10x = 39$

- When geometric methods were used to solve quadratic equations, Eqns were always written such that there were no negative coefficients, as below:

$x^2 + bx = c$; $x^2 = bx + c$, $x^2 + c = bx$

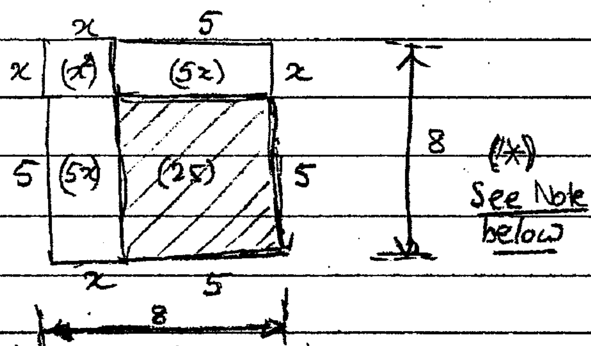
(with 'b' & 'c' always positive!)

Geometrically, we have

$$\overset{x}{x\,\boxed{(x^2)}} + \overset{10}{\boxed{(10x)}}\,x = 39$$

- Let us divide $10x$ as 2 strips of $\frac{1}{2}(10x) = 5x$

Rearranging,



(5×5) completes the square.

Refering to our equation

$$x^2 + 10x + \underbrace{25}_{(Add)} = 39 + \underbrace{25}_{(Add)}$$
$$= 64$$

Note

(*) The new "completed" square, has side length of $\sqrt{64} = 8$.

$\therefore (x+5) = 8 \quad \therefore x = 8 - 5 = 3$

Note: This avoids dealing with negative solutions!!

- In 1545, Jerome Cardano (Italian)(1501-1576) developed a method for solving the 'real' root of a cubic equation

- when solving the eqn

$$x^3 = 15x + 4$$

he ended up with the following

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

- Cardano did NOT know how to handle $\sqrt{-121}$, even though he knew that the final result is positive!
- About 10 years later, Rafael Bombelli solved the problem,

- Let us consider the simplified problem below, to understand the method!

Ex2. Calculate $\sqrt{2 + \sqrt{-66}} + \sqrt{2 - \sqrt{-66}}$

Say, $x = \sqrt{2 + \sqrt{-66}} + \sqrt{2 - \sqrt{-66}}$

$$\therefore x^2 = \left(\sqrt{2+\sqrt{-66}} + \sqrt{2 - \sqrt{-66}}\right)^2$$

$$= \left(2 + \cancel{\sqrt{-66}}\right) + \left(2 - \cancel{\sqrt{-66}}\right) + 2.\left(\sqrt{2+\sqrt{-66}}\right)\left(\sqrt{2-\sqrt{-66}}\right)$$

$$= 4 + 2\left\{\sqrt{(2+\sqrt{-66})(2-\sqrt{-66})}\right.$$

$$= 4 + 2\left\{\sqrt{4 + \cancel{2\sqrt{-66}} - \cancel{2\sqrt{-66}} - \sqrt{-66}\sqrt{-66}}\right\}$$

$$= 4 + 2\left\{\sqrt{4+66}\right\} = 20.733$$

$$\therefore x = \sqrt{20.733} = \underline{\underline{4.55}}$$