{Week 6 / No Class}

U3A Maths

~~10~~ 17-Mar-2023 Term 1/~~Week 6~~ Week 7

Complex Numbers
Euler's Formula & Identity

Review

- Euler showed that multiplication by $i = \sqrt{-1}$ as a "rotation" of the value by 90°, based on the concept that multiplication by $i^2 = -1$ is a rotation of the value by 180°.
- Gauss formalised the complex number form as
  $$Z = (a + bi)$$
  This is the Cartesean form of complex number.

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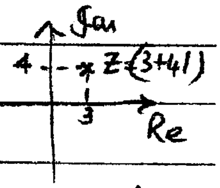
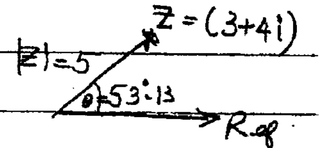
- Argand introduced the representation of complex numbers as a "point" on a two dimensional graph (Cartesean form)



- A complex number can also be represented in the Polar form

Magnitude
$$|Z| = \sqrt{a^2 + b^2}$$
Angle to Ref. axis
$$\theta = \tan^{-1}(b/a)$$



- We use Cartesean or Polar form, as per convenience.

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- Euler also established a formula which related 'i' ~~which related~~ to exponential (e^x) and trigonometric functions $\sin(x)$ & $\cos(x)$.
- Note that the value of exponential constant ($e = 2.718$) was first discovered by Jacob Bernoulli while calculating compound interest!
- Euler later formalised it and gave it the name "Exponential" with symbol 'e'.

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- From Taylor's series, we have,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

Hence,

$$e^{ix} = 1 + (ix) + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \dots$$

$$= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - + \dots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \dots\right)$$

$$+ i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \dots\right)$$

$$= \cos(x) + i\sin(x)$$

- The above is the famous Euler's formula!

$$\boxed{e^{ix} = \cos(x) + i\sin(x)}$$

- Also, we have

$$Ae^{ix} = A\left[\cos(x) + i\sin(x)\right]$$

where A is any constant — normally a real number.

- We now have an <u>algebraic</u> <u>expression</u> which is related to a <u>trigonometric</u> <u>expression</u>.

- Note also that

$$\frac{d}{dx}(e^x) = e^x \quad \& \quad \int e^x dx = e^x$$

- Hence e^x is a convenient function for differentiation and integration!

- Finally, we have

$$\text{Real Part of } \left[Ae^{ix}\right] = A\cos(x)$$
$$\text{Im. Part of } \left[Ae^{ix}\right] = A\sin(x)$$

- Cartesean form of Ae^{ix} is $\cos x + i\sin x$ (where 'x' is normally in <u>Radians</u>)

- So, we have 3 forms for representing complex numbers

(1) $Z = (a+bi)$ — Cartesean form

(2) $Z = |Z| \angle\theta$ — Polar form

where $|Z| = \sqrt{a^2+b^2}$

$\theta = \tan^{-1}(b/a)$

(3) $Z = |Z|e^{i\theta}$ — Exponential Form

$= |Z|(\cos\theta + i\sin\theta)$

Form (1) is useful for Addition/Subt.
Form (2) is useful for Mult./Division
Form (3) is useful for differentiation/Integration

- <u>Euler's Identity</u>

- A <u>particular</u> case of Euler's formula is called the <u>Euler's Identity</u>.

- When $x = \pi$ radians;

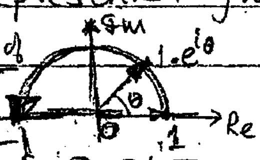
$$e^{i\pi} = \cos\pi + i\sin\pi = -1 + 0 = -1$$

0 - Non-existence
1 - Existence

$$\therefore \quad \boxed{e^{i\pi} + 1 = 0}$$

- Note: e, i & π are transcendental Nos.!
- It is known as the most <u>beautiful</u> <u>equation</u> in maths. It can be represented graphically as below

Plot of $e^{i\theta}$



for $\theta = 0$ to π

- <u>This identity holds the secret to the universe!</u>