U3A Maths

26 - May - 2023 Term 2 / Week 5

Complex Number Multiplication

Review

- Complex addition / subtraction is straight forward in cartesian form.

$$(a+bi) + (c+di) = (a+c) + i(b+d)$$
$$(a+bi) - (c+di) = (a-c) + i(b-d)$$

- For polar form, convert polar values into cartesian form prior to addition/subtraction

$$Z = R\angle\theta = (R\cos\theta + iR\sin\theta)$$

- Complex multiplication in cartesian form is straigh forward.

-2-

$$(a+bi) \times (c+di)$$
$$= (ac-bd) + i(bc+ad)$$

- Complex multiplication in polar form is much simpler!

$$R_1\angle\theta_1 \times R_2\angle\theta_2 = R_1R_2\angle(\theta_1+\theta_2)$$

- Complex division in cartesian form is cumbersome, and requires the use of "conjugate of the denominator".

$$\frac{(a+bi)}{(c+di)} = \frac{(a+bi)}{(c+di)} \times \frac{(c-di)}{(c-di)}$$
$$= \frac{(a+bi)(c-di)}{(c^2+d^2)}$$

-3-

- Complex division in polar form is much simpler:

$$\frac{R_1\angle\theta_1}{R_2\angle\theta_2} = \left(\frac{R_1}{R_2}\right)\angle\theta_1-\theta_2$$

Note: $R\angle\theta$ is synonymous with $Re^{i\theta}$

Multiplication by an unit vector

- Multiplication by an unit vector with a given angle provides some interesting results.

Unit vector $\Rightarrow$ $1\angle\theta$

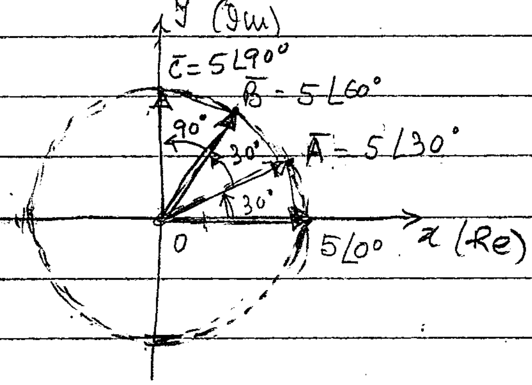
-4-

Ex Multiply $5\angle 0°$ & $1\angle 30°$

let

$$\bar{A} = 5\angle 0° \times 1\angle 30° = 5\times 1\angle 0+30°$$
$$= 5\angle 30°$$

Let us plot the result!



- Successive multiplications will result in points B, C etc
- Hence, it is possible to generate various shapes, such as, square, hexagon, octogon, circle etc!

- Complex multiplication can be used to produce impressive computer graphics and graphic designs.

Complex Roots

Let us re-visit the solution of equations with "imaginary" or "complex" roots

Ex: Solve $x^3-1=0$

By inspection, $x=1$ is a root.

$\therefore\ (x-1)=0$

Dividing: $(x^3-1)\div(x-1)$
$=(x^2+x+1)$

$$\begin{array}{r|l} & x^2+x+1 \\ \hline (x-1) & x^3-1 \\ & x^3-x^2 \\ \hline & 0+x^2-1 \\ & x^2-x \\ \hline & 0+x-1 \\ & x-1 \\ \hline & 0+0 \end{array}$$

Let us now find the roots $(x^2+x+1)=0$

we get

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-1\pm\sqrt{1-4}}{2}$$

$$= -\tfrac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

The roots are:

$x = \underline{1},\ \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) \& \left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$

In complex form $\Rightarrow$ $1\angle 0^\circ$, $1\angle 120^\circ$, $1\angle -120^\circ$

- Obviously, the complex (imaginary!) roots does not make much sense!

- Let us now explore the roots using complex arithmetic.

we have $x^3=1$

or $x.x.x=1$

In complex form, we have

$\bar{x}\cdot\bar{x}\cdot\bar{x}=1$

- Where "$\bar{x}$" is a complex number & let us think in polar form as it involves multiplication

- $\bar{x}_1 = 1\angle 0^\circ$ is a trivial solution
- Also, the angles add in complex multiplication, hence 3 angles adding upto 360° is also a solution. Hence, another solution is $1\angle 360^\circ/3 = 1\angle 120^\circ$

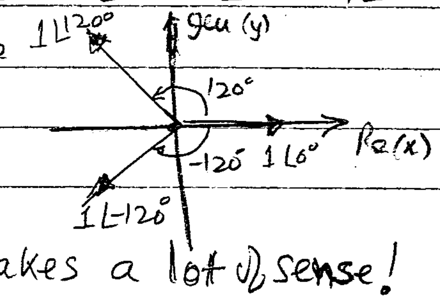
i.e. $1\angle 120^\circ \times 1\angle 120^\circ \times 1\angle 120^\circ = 1\angle 360^\circ = 1\angle 0^\circ$

The 3 roots are

$\bar{x}_1 = 1\angle 0^\circ$

$\bar{x}_2 = 1\angle 120^\circ$

$\bar{x}_3 = 1\angle -120^\circ$



- This result makes a lot of sense!