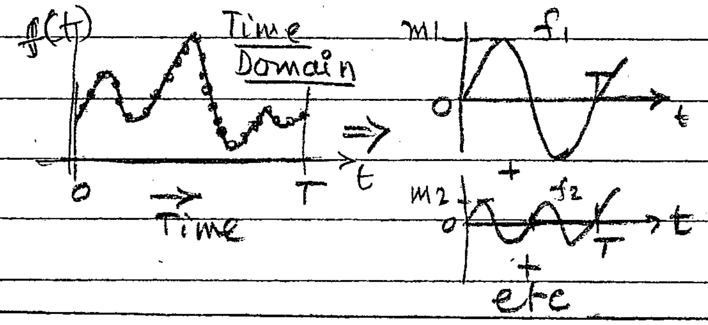
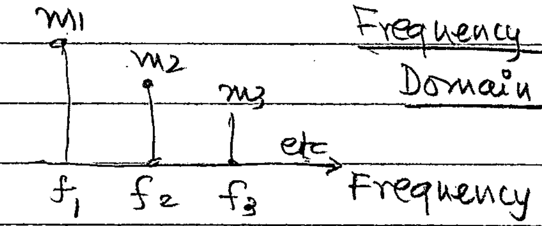
U3A Maths

30-Jun-2023 Term 2 / Week 10

Discrete Fourier Transform (DFT)

- Review
  - DFT is the most used mathematical equation in the modern world.
  - The Fourier Transform resolves any given function form into sinusoidal waves





- Frequency components provide a more efficient (up to about 100:1) way to store / transmit data.
- The equation for Fourier Transform is as below:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt \quad \text{—(1)}$$

(where $\omega = 2\pi f$ & $f = \frac{1}{T}$ & $i = \sqrt{-1}$)

Inverse Fourier Transform

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)\, e^{+i\omega t}\, d\omega \quad \text{—(2)}$$

- Fourier Transform equations above are applicable when the data is given as an algebraic function.
- However, when (digital) computers are used, it is common to express the data as a set of numerical values.
- In such a case, Discrete Fourier Transform equations are used — which are much simpler! The integration is expressed as summation!!

- Discrete Fourier Transform (DFT) and Inverse DFT equations are as below:
  - DFT Equation (Compare with eqn (1))

$$X(K) = \sum_{n=0}^{N-1} x(n) \cdot e^{-i\frac{2\pi K n}{T}}$$

(for $K = 0$ to $N-1$)

- Note the $X(K)$ and $x(n)$ are used instead of $F(\omega)$ & $f(t)$ — as they are a set of numeric values rather than functions.
- Note that 'K' still represents the 'frequency' of the sinusoidal components.

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- Inverse DFT equation is

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) \cdot e^{+i\frac{2\pi nk}{N}}$$

(for $n = 0$ to $N-1$)

- Note that in DFT, the maximum possible frequencies are "N" — which is the maximum data points available.
- More data points are required, if a higher frequency spread is required.

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- Let us consider a simple Example

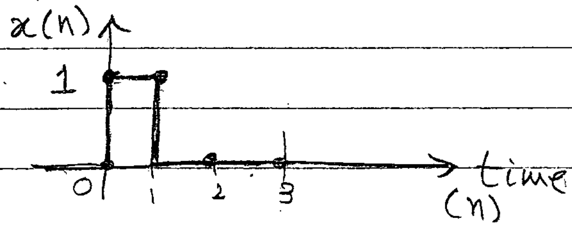
Ex.1 Find the DFT components for the following data.

$$x(n) = \{1, 1, 0, 0\}.$$

where $n = 0$ to 3

Note That $N = 4$ & $(N-1) = 3$

Graphically the data can be represented as below:



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Using DFT Equation:

K=0

$$X(0) = \sum_{n=0}^{3} x(n) \cdot e^{-i\frac{2\pi \cdot 0 \cdot n}{4}}$$

$$= x(0) \cdot e^{-0} + x(1)e^{-0} + x(2)e^{-0} + x(3)e^{-0}$$

$$= 1 + 1 + 0 + 0 = \underline{2}$$

K=1

$$X(1) = \sum_{n=0}^{3} x(n) \cdot e^{-i\frac{2\pi \cdot 1 \cdot n}{4}}$$

$$= x(0) \cdot e^{-i\frac{2\pi \cdot 0}{4}} + x(1)e^{-i\frac{2\pi \cdot 1}{4}} + x(2)e^{-i\frac{2\pi \cdot 2}{4}} + x(3)e^{-i\frac{2\pi 3}{4}}$$

$$= 1 \cdot e^{0} + 1\, e^{-i\pi/2} + 0 + 0$$

$$= 1 + 1\,(\cos\pi/2 - i\sin\tfrac{\pi}{2})$$

$$= \underline{(1-i)}$$

Similarly, we can calculate $X(2) = \underline{0}$, $X(3) = (1+i)$

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Ex.2 Find inverse DFT for the above Example i.e.,

$$X(k) = \{2, (1-i), 0, (1+i)\}.$$

Note that $N=4$, $(N-1)=3$

Using Inverse DFT equation:

$$x(0) = \frac{1}{4}\sum_{k=0}^{3} X(k) \cdot e^{+i\frac{2\pi \cdot 0 \cdot k}{4}}$$

$$= \frac{1}{4}\left(X(0) \cdot \cancel{e^{+i\frac{2\pi \cdot 0 \cdot 0}{4}}} + X(1)\cancel{e^{+i\frac{2\pi \cdot 1 \cdot 0}{4}}} + X(2)\cancel{e^{+i\frac{2\pi \cdot 2 \cdot 0}{4}}} + X(3)\cancel{e^{i\frac{2\pi \cdot 3 \cdot 0}{4}}}\right)$$

$$= \frac{1}{4}(X(0) + X(1) + X(2) + X(3))$$

$$= \frac{1}{4}(2 + (1-\cancel{i}) + 0 + (1+\cancel{i}))$$

$$= \frac{1}{4}(4) = \underline{1}$$

Similarly, we can calculate:

$x(1) = 1$, $x(2) = 0$, $x(3) = 0$

– We get back original data!