U3A Maths

9-Jun-2023 Term 2 / Week 7

No class on week 6

Complex Roots

Review

- Roots of Real function

$$f(x) = x^3 - 1 = 0$$
$$= (x-1)(x^2+x+1) = 0$$

$\therefore$ Roots are: $x = 1$; $x = \left(-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i\right)$

- Roots of Corresponding Complex fn.

$$f(\bar{z}) = \bar{z}^3 - 1 = 0$$

we can write $\bar{z}^3 = 1$

or $\bar{z}.\bar{z}.\bar{z} = 1$

By inspection roots are:

$$\bar{z}_1 = 1\angle 0^\circ \;;\; \bar{z}_2 = 1\angle 120^\circ \;;\; \bar{z}_3 = 1\angle -120^\circ = 1\angle +240^\circ$$

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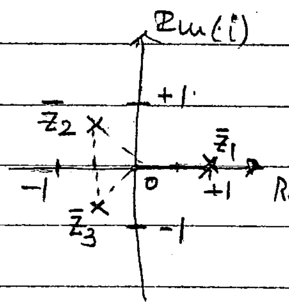
Roots in Cartesian form

$$\bar{z}_1 = (1 + i0)$$
$$\bar{z}_2 = 1(\cos 120^\circ + i\sin 120^\circ) = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right)$$
$$\bar{z}_3 = 1(\cos 240 + i\sin 240) = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\right)$$

Note: In complex form, the roots make more sense!



- Complex roots are used extensively in the analysis of stability of control systems!

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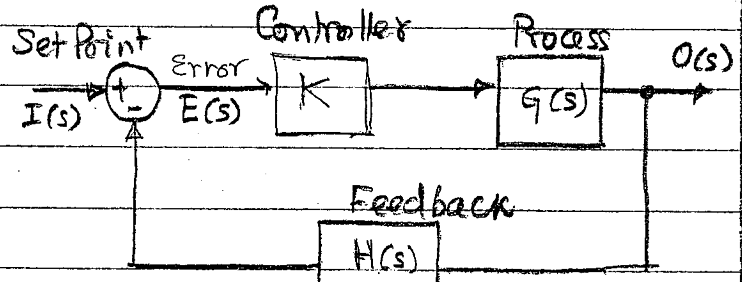
Simple Control system



{where 's' is Laplace Transform Variable}

We have:

$$E(s) = I(s) - O(s).H(s) \quad -(1)$$

& $$O(s) = E(s)\,[K.G(s)] \quad -(2)$$

Substituting for E(s) in (2) from (1)

$$O(s) = [I(s) - O(s).H(s)]\,[K.G(s)]$$

Simplifying, we get

$$\frac{O(s)}{I(s)} = \frac{K.H(s).G(s)}{1 + K.H(s)G(s)} \quad -(3)$$

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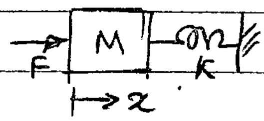
- For simplicity let us assume $H(s) = 1$
- In practice, the "process" is a differential equation, for Ex, for Mass-spring system

$$F = M.\underbrace{\frac{d^2x}{dt^2}}_{\text{Accelaration}} + kx.$$



- The differential equations are converted into algebraic equations (with 's' varible) using Laplace transformation, to solve differential Eqns.
- Let us assume that the process is

$$G(s) = \frac{1}{s(s+3)}$$

after Laplace Transformation.

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We now have:

$$\frac{O(s)}{I(s)} = \frac{K \cdot G(s)}{1+KG(s)} = \frac{K/s(s+3)}{1+K/s(s+3)}$$

$$= \frac{K}{s(s+3)+K} = \boxed{\frac{K}{s^2+3s+K}}$$

Let us look at the denominator

Case (a) K=1

$$s^2+3s+1$$

The roots are:

$$s = \frac{-3 \pm \sqrt{3^2-4}}{2} = -1.5 \pm 1.12$$

$$= -2.7 \text{ or } -0.3$$

Case (b) $K=10 \Rightarrow s^2+3s+10$

The roots are

$$s = \frac{-3 \pm \sqrt{3^2-4\times 10}}{2} = -1.5 \pm i\,2.78$$

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Case (c) $K=50 \Rightarrow s^2+3s+50$

The roots are:

$$s = \frac{-3 \pm \sqrt{9-4\times 50}}{2} = -1.5 \pm i\,6.91$$

Control System Response

- The Eqn $\frac{O(s)}{I(s)} = \frac{K}{s^2+3s+K}$ can be solved using Laplace transformation for K = 1, 10 & 50.

- The results after converting back to "time domain" are as shown below:

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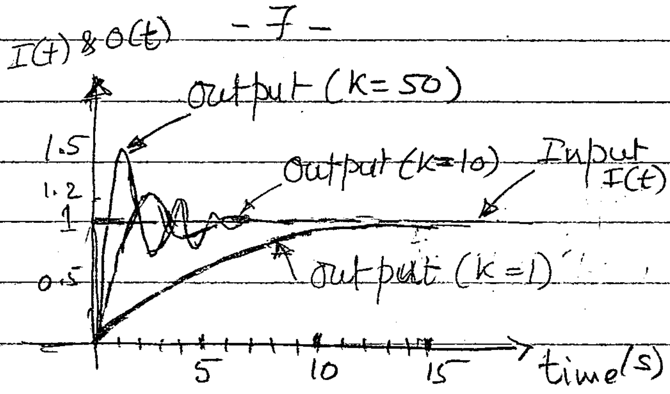

- The above plots show the control/process responses for various values of K

- As per the plots, the ideal response is closer to K=10;

- The above conclusions can also be made using the plots of Complex Roots, without solving diffl. eqns!!

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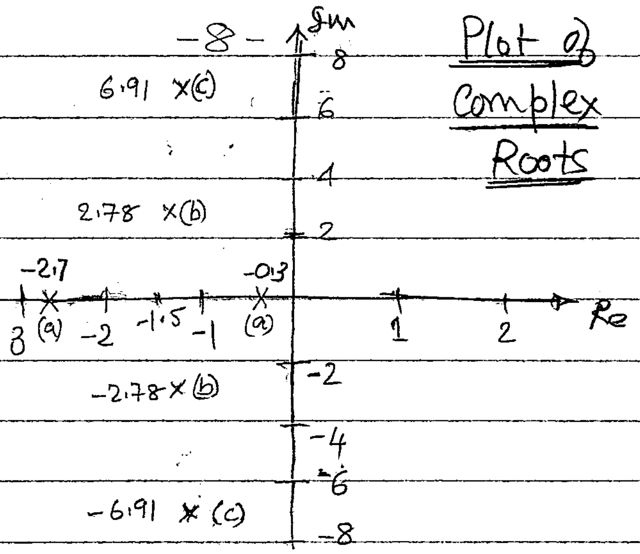

Plot of Complex Roots

- The system is stable if the real part of the roots are negative

- The system is oscillatory if imaginary part is present. Higher the imaginary magnitude, oscillation is higher

- The system is unstable, if the real part is positive