U3A Maths

21-JUL-2023 Term 3 / Week 1

Discrete Fourier Transform (DFT)

Complex Numbers Overview

- Square root of a negative number was first encountered while solving for the "real" root of a cubic equation.
- The problem was circumvented by representing square root of the negative number as below.

Ex: $\sqrt{-121} = \sqrt{(-1)(121)}$
$= (\sqrt{-1})(\sqrt{121})$
(where $i = \sqrt{-1}$) $= i\,11$ or $11\,i$

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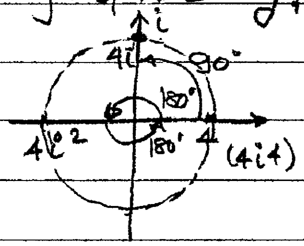
- The calculations were done by treating "i" as an algebraic variable and 'i' was cancelled out in the final result, leaving a real number as the result.
- Since, $i = \sqrt{-1}$ $\therefore$ $i^2 = -1$

For Ex. $4i^2 = -4$
& $4 \cdot i^4 = 4 \cdot i^2 \cdot i^2 = +4$

- When above is represented graphically:



Hence, $4i^2 \Rightarrow 180°$ shift
$\therefore$ $4i$ represents $90°$ shift!

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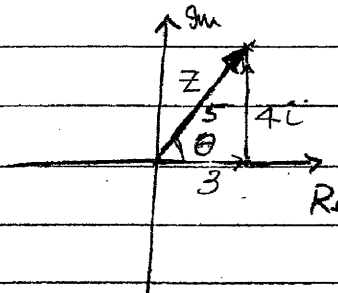
- Hence a value such as $(3 + 4i)$ represents vector!



Cartesian Form
$z = (3 + 4i)$

Polar Form
$z = \sqrt{3^2+4^2} \,\angle\, \tan^{-1}(\tfrac{4}{3})$
$= 5 \angle 53.13°$

- We can write $z = (3 + 4i)$ as a "complex" or "a two dimensional" number.
- We can show that "complex numbers" are the most general form of number system, since any arithmetic operation with complex numbers result in a complex number!

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- Euler helped to formalise the polar form as an algebraic expression.

$$z = A e^{i\theta}$$
$$= A(\cos\theta + i\sin\theta)$$

For our Ex:
$$z = 5e^{i\,53.13°}$$
$$= 5(\cos 53.13° + i \sin 53.13°)$$
$$= (3 + i4)$$

- All arithmetic operations on complex numbers can be done using the usual algebraic methods!

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- Fourier Transform resolves any given function into its sinusoidal component waves!
- Discrete Fourier Transform (DFT) is used for a set of discrete values!
- DFT equation is as below:

For a given set of 'x'

$$x = \{x(0), x(1), \cdots x(N-1)\}$$

the D.F.T. components $F(k)$ are as below: (for $K=0$ to $(N-1)$)

$$F(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-i\frac{2\pi k n}{N}}$$

where "N" is number of values and corresponds total "time period" of one cycle or 2π radians.

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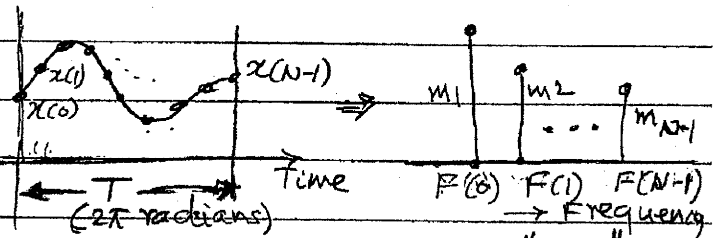
$F(k)$ - D.F.T. components are sinusoids, since $e^{i\theta} = (\cos\theta + i\sin\theta)$!



Note
- $x(0), x(1) \cdots$ are "real" values
- $F(0), F(1) \cdots$ are "complex" values. (normally Magnitudes are plotted!)

Home Work (solution)

Find the D.F.T. of

$$x = \{1, 1, 1, 1\}$$

we have: $N=4$ & $(N-1)=3$

$x(0)=1$, $x(1)=1$, $x(2)=1$, $x(3)=1$

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Using D.F.T. equation:

For $k=0$:

$$F(0) = \sum_{n=0}^{3} x(n) e^{-i\frac{2\pi \cdot 0 \cdot n}{4}}$$

$$= x(0) \cdot e^{-i0} + x(1) e^{-i0} + x(2) e^{-i0} + x(3) e^{-i0}$$

$$= 1+1+1+1 = \underline{4}$$

For $k=1$

$$F(1) = \sum_{n=0}^{3} x(n) \cdot e^{-i\frac{2\pi \cdot 1 \cdot n}{4}}$$

$$= x(0) e^{-i\frac{2\pi \cdot 1 \cdot 0}{4}} + x(1) e^{-i\frac{2\pi \cdot 1 \cdot 1}{4}} + x(2) \cdot e^{-i\frac{2\pi \cdot 1 \cdot 2}{4}} + x(3) e^{-i\frac{2\pi \cdot 1 \cdot 3}{4}}$$

$$= 1 \cdot e^{-i0} + 1 \cdot e^{-i\pi/2} + 1 \cdot e^{-i\pi} + 1 \cdot e^{-i\frac{3\pi}{2}}$$

$$= \underline{0}$$

Similarly, we can show that

$F(2)=0$ & $F(3)=0$

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Inverse DFT equation

$$x(n) = \frac{1}{N} \sum_{K=0}^{(N-1)} F(K) \cdot e^{+i\frac{2\pi n k}{N}}$$

(for $n=0$ to $N-1$)

For our Ex: $N=4$ & $(N-1)=3$

$F(0)=1$, $F(1)=0$, $F(2)=0$, $F(3)=0$

for $n=0$

$$x(0) = \frac{1}{4} \sum_{K=0}^{3} F(K) e^{i\frac{2\pi \cdot 0 \cdot k}{4}}$$

$$= \frac{1}{4} \{F(0) \cdot e^{i0} + F(1) \cdot e^{i0} + F(2) e^{i0} + F(3) e^{i0}\}$$

$$= \frac{1}{4}\{4+0+0+0\} = \frac{4}{4} = \underline{\underline{1}}$$

Similarly, we can show

$x(1)=1$, $x(2)=1$ & $x(3)=1$