U3A Maths

21-May-2024 Term 2 / Week 4

Complex Numbers
(An Alternative View)

Review

- A Set of Positive numbers (including fractions) is closed for Addition, Multiplication, Division & square root but not necessarily for subtraction.

$2+3=5$ ✓
$2\times 3=6$ ✓
$3/2=1.5$ ✓
$\sqrt{2}=1.414$ ✓
$3-2=1$ ✓ $2-3=-1$ ✗

$\{0, 1, 2, 3, 4, 5, \ldots$
$1/2, 1/4, 3/8, 1/3, \ldots$
$\pi, e, \ldots\}$

- A set of Real numbers (includes negative numbers) is closed for addition, subtraction, multiplication, division but not necessarily for 'square root'

$2-3=-1$ ✓
$\sqrt{4}=\pm 2$ ✓
$\sqrt{-4}=?$ ✗

$\{0, 1, 2, 3, 4, 5, \ldots$
$1/2, 1/4, 1/3, 3/8, 3/6, \ldots$
$e, \pi,$
$-1, -2, -3, \ldots$
$-1/2, -1/4, \ldots\}$

- The above set is 'closed' if we include $\sqrt{-1}$ in our set.

$\therefore \sqrt{-4}=\sqrt{(-1)(4)}=\sqrt{-1}\cdot\sqrt{4}$
$= i2$ (where $i=\sqrt{-1}$)

- Negative numbers and square root of negative numbers was introduced by Cardano in 1545 AD. in the Book: Ars Magna (Great Art!)

- Rene Des Cartes (1596-1650 AD) cynically coined the term "Imaginary" number for $\sqrt{-1}$, as he could not represent $\sqrt{-1}$ geometrically on a linear Scale!

- Leonhard Euler (1707-1783 AD) (Swiss mathematician who spent most of his adult life in Germany & Russia) made $\sqrt{-1}$ relevant & useful in practice.

- Euler was also responsible for formalising the use of the exponential 'e' $=2.718\ldots$ as a part of compound interest calculations.

- He developed a relationship between 'e' & 'i' as below:

$$e^{i\theta}=\cos(\theta)+i\sin(\theta)$$

- The above equation not only makes 'i' mathematically relevant but it also formally defines a complex number. For Ex.

(a) $5e^{i0^\circ} = 5(\cos 0^\circ + i\sin 0^\circ) = 5$

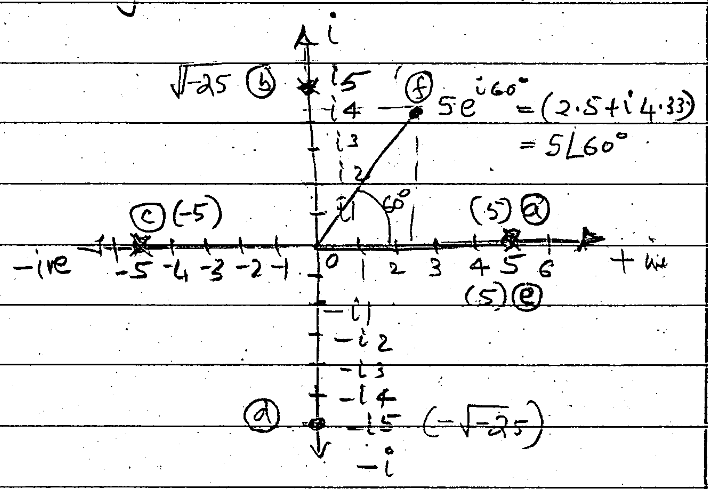
(b) $5e^{i90^\circ} = 5(\cos 90^\circ + i\sin 90^\circ) = i5 = \sqrt{-25}$

(c) $5e^{i\pi} = 5(\cos\pi + i\sin\pi) = -5$

(d) $5e^{i270^\circ} = 5e^{-i90^\circ} = -i5 = -\sqrt{-25}$

(e) $5e^{2\pi} = 5e^{360^\circ} = 5e^{0} = 5$

Plotting the above values:



In general, we can write

$$5e^{i\theta} = 5(\cos\theta + i\sin\theta).$$

for $\theta = 60^\circ$ (point (f) on the Figure)

$$5e^{i60^\circ} = 5(\cos 60^\circ + i\sin 60^\circ)$$
$$= 5(0.5 + i\,0.866)$$
$$= (2.5 + i4.333)$$
$$= 5\angle 60^\circ$$

- Hence, when θ varies from 0 to 360°, the value or magnitude of 5 moves along a circle.

- Hence, we essentially have a point in two dimensions.

- In other words, the traditional number system is a "one-dimensional number, whereas a complex number is a two-dimensional number

- Finally, we can show that a complex number is a closed set for all arithmetic operations, namely addition, subtraction, multiplication, division and square root!

- Let us find square root of a complex number.

Given $A = 4e^{i60}$ Find $\sqrt{A}$

$$\sqrt{A} = (A)^{1/2} = (4e^{i60^\circ})^{1/2}$$
$$= (4)^{1/2}\,(e^{i60^\circ})^{1/2}$$
$$= \sqrt{4}\cdot(e^{i60^\circ})^{1/2}$$

[we have $(a^x)^{1/2} = a^{x/2}$]

$$\therefore \sqrt{A} = \sqrt{4}\cdot e^{i60/2} = 2\cdot e^{i30^\circ}$$
$$= 2(\cos 30 + i\sin 30^\circ)$$

Result is a complex No. $= (1.732 + i\,1)$

Square Root is closed!

Home work Given $A = -4e^{i60^\circ}$ & $B = -4e^{-i90}$

Find $\sqrt{A}$, $\sqrt{B}$ & $\sqrt{A\cdot B}$