U3A Maths

28-May-2024 Term 2 / Week 5

Complex Numbers
(An Alternative View)

- The "classical" number system is not "closed" (complete) since it does not include square root of a negative numbers.
- The above problem is solved by included $\sqrt{-1}$ (denoted by symbol "i") as a part of the number system. For Ex:

$$\sqrt{-4} = \sqrt{(4)(-1)} = \sqrt{4}\cdot\sqrt{-1} = 2i$$

- A general for complex numbers was developed ("invented"!) by Euler.
- Euler's form provided for representation of a general purpose number, which includes "i" ($\sqrt{-1}$)

$$Ae^{i\theta} = A\cos\theta + iA\sin\theta$$

where,
θ is an Angle w.r.t. a "Reference Axis", which is normally is the traditional 'x' axis in positive direction.

- Any given number can be written in the complex form:

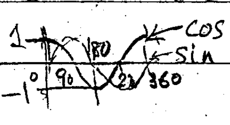
Ex:

(a) $4 \Rightarrow 4e^{i0} = 4(\cos 0° + i\sin 0°)$
$= 4(1 + i0) = 4$

(b) $-5 \Rightarrow -5e^{i0°}$

we prefer to keep this as a positive value. We call it the "Magnitude".
Hence, a complex number has a "Magnitude" and an "Angle"
∴ we have

$-5 \Rightarrow -5e^{i0°} = 5e^{i180°}$



$= 5(\cos 180° + i\sin 180°)$
$= 5(-1 + i0) = -5$!

(c) $\sqrt{-4} \Rightarrow (\sqrt{4})(\sqrt{-1}) = 2i$
$= 2e^{i90°}$
$= 2(\cos 90° + i\sin 90°)$
$= 2(0 + i\cdot 1)$
$= 2i$ or $\sqrt{-4}$!

- We can perform all arithmetic operations using the complex form.
- Using the complex form solves some of the mysteries of operations using negative numbers

For Ex: $-2 \times -2 = +4$
$\sqrt{-4} = 2i$

Homework

Given $A = -4\,e^{i60}$ &

$B = -4\,e^{-90^\circ}$

Calculate

(a) $\sqrt{A}$

we have

$$A = -4\,e^{i60^\circ}$$

$$= 4\,e^{i\,60^\circ+180^\circ}$$

↑ Expressing as Magnitude

$$= 4\,e^{i240^\circ}$$

let

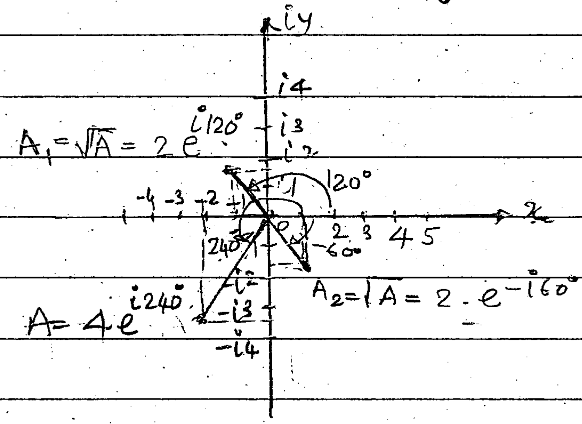
$$A_1 = \sqrt{A} = (A)^{1/2} = (4\,e^{i240^\circ})^{1/2}$$

$$= 4^{1/2}.\,e^{i240/2}$$

$$= (-1+i1.732) = 2.\,e^{i120^\circ}$$

$$= 2(\cos 120^\circ + i\sin 120^\circ)$$

Expressing graphically



$\sqrt{A}$ = square Root of Magnitude at half the Angle!

• Note: we can get another answer by setting $A = 4.\,e^{i(60-180)}$

$= 4\,e^{-i120^\circ}$ (same as $4e^{i240^\circ}$)!

But $\sqrt{A} = 2\,e^{-i60^\circ}$ ⇒ say A_2

$$A_2 = \sqrt{A} = 2\,[\cos(-60^\circ) - i\sin(-60^\circ)]$$

$$= 1 -$$

• Note that A_1 & A_2 are two roots of A

Note: $A_1 = (-1+i1.732) = 2e^{i120^\circ}$

$A_2 = -(A_1)$

$= (1 - i1.732) = 2e^{-60^\circ}$

• Considering the Example problem

$$\sqrt{-4} = \pm 2i$$

We have $\sqrt{-4} = (4\,e^{i180})^{1/2}$

$= 2.\,e^{i90} = 2i$

Also $\sqrt{-4} = (4\,e^{-i180^\circ})^{1/2}$

$= 2\,e^{-i90}$

$= -2i$

(b) $B = -4\,e^{-i90^\circ}$

$= 4.\,e^{i180}.\,e^{-i90}$

$= 4\,e^{i(180^\circ-90^\circ)}$

$= 4\,e^{i90^\circ}$

$\therefore \sqrt{B} = (4e^{i90})^{1/2} = 2.\,e^{i45}$

Note: $\sqrt{B}$ is also $= 2\,e^{i135}$ (check)

(c) $A = -4\,e^{i60} = 4e^{i240}$

$B = -4e^{-90^\circ} = 4\,e^{i90}$

$\therefore A \times B = 4\,e^{i240^\circ} \times 4\,e^{i90^\circ}$

$= 16\,e^{i330^\circ}$ or $16\,e^{-i30^\circ}$

$\therefore \sqrt{A \cdot B} = (16\,e^{-i30^\circ})^{1/2}$

$= 4\,e^{-i15^\circ}$