U3A Maths

10-Jun-2024 Term 2/Week 7

Complex Number - Power & Roots

Review Most Beautiful Eqn

$$\underbrace{e^{i\pi}}_{\text{Transcendence}} + \underbrace{1}_{\text{Existence}} = \underbrace{0}_{\text{Null or Non-Existence or Blackhole!!}}$$

- Powers of Complex Number

Let $z = 3 + 4i$

we have: $z^2 = (3+4i)^2$
$= (3+4i)(3+4i)$
$= 9 + 24i + 16i^2 = 9 + 24i - 16$
$= (-7 + 24i) = 25\angle 106^\circ.26$

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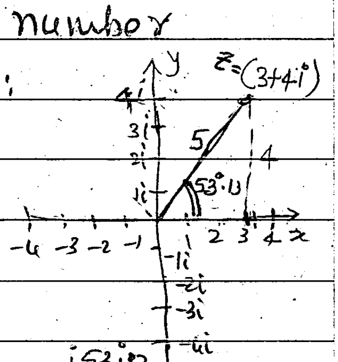
- It is easier to calculate the powers of complex number in Euler's form:



Ex:
$z = (3 + 4i)$
$= \sqrt{3^2+4^2} \,/\, \tan^{-1}\frac{4}{3}$
$= 5\angle +53^\circ.13 = 5e^{i53^\circ.13}$

$z^2 = (5e^{i53^\circ.13})^2 = 5^2 (e^{i53^\circ.13})^2$
$= 25\, e^{i\,2\times 53^\circ.13}$
$= 25\, e^{i106^\circ.26} = 25\angle 106^\circ.26$

- The beauty of Euler's form is that, we can calculate any given power!

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Ex: $z = (3+4i)^8$
This calculation is laborious!

However, in Euler's form:
$z^8 = (5e^{i53.13})^8$
$= 5^8 \cdot e^{i53^\circ.13\times 8}$
$= 5^8 \cdot e^{i425^\circ.04}$
$= 5^8 \cdot e^{i(425^\circ.04 - 360^\circ)}$
$= 5^8 \cdot e^{i65^\circ.04}$

- In general we can write

$$z^n = (r.e^{i\theta})^n = r^n . e^{in\theta}$$
$$= r^n(\cos n\theta + i\sin n\theta)$$

This is called
De Moivre's Theorem
(Proved by Mathematical Induction)

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- The interesting fact is that De Moivre's Theorem was developed before Euler's Equation!
[Euler 1707–1783]

- Abraham de Moivre (1667–1754)
  - Born in France in "Huguenot" (protestant) family.
  - Moved to England in 1685 (due to persecution)
  - Friend of Newton & Halley.
  - worked in Complex numbers, trigonometry & probability theory!

- The beauty of De Moivre's theorem is that it can be extended to Roots of complex numbers!

- We know that square root of 4 has two roots: +2 & −2
- We have $\sqrt[3]{8} = 2$

$\therefore \sqrt[3]{8}$ has three roots; what are the other two roots?

It is not -2 since $-2 \times -2 \times -2 = -8$

or $2i$ since $2i \times 2i \times 2i = -8i$

- Let us consider a simpler example.

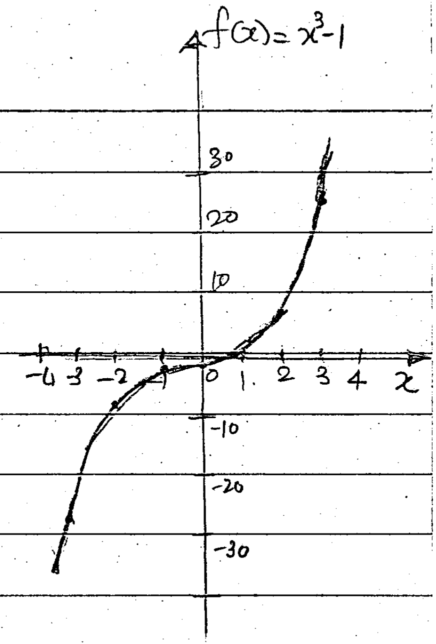
Solve $x^3 - 1 = 0$

ie, $x^3 = 1$

We know that $x = 1$ is a root.

- The graphical method does not help to find the other roots!

x	$f(x) = x^3 - 1$
0	−1
1	0
2	7
3	26
−1	−2
−2	−9
−3	−28



- Let us use a complex variable to solve the above problem.

Let $z^3 = 1$ where $z = re^{i\theta}$

$z^3 = r^3 e^{i3\theta} = r^3 (\cos(3\theta) + i\sin(3\theta)) = 1$

Since $r = 1$, the above eqn is valid when,

$\cos(3\theta) = 1$ and $\sin(3\theta) = 0$

The above is true when

$3\theta = 2\pi k$ where $k = 0, 1, 2, \ldots$ (multiples of 2π or $360°$)

$\therefore \theta = \frac{2\pi k}{3}$ where $k = 0, 1 \& 2$

(Since we have only 3 roots)

Let us say x_0, x_1 & x_2 are the roots

Hence,

$$x_0 = \cos\left(\frac{2\pi \times 0}{3}\right) + i\sin\left(\frac{2\pi \times 0}{3}\right) = 1$$

$$x_1 = \cos\left(\frac{2\pi \times 1}{3}\right) + i\sin\left(\frac{2\pi \times 1}{3}\right)$$

$$= (-0.5 + i\,0.866) \text{ or } 1.e^{i\frac{2\pi}{3}}$$

$$x_2 = \cos\left(\frac{2\pi \times 2}{3}\right) + i\sin\left(\frac{2\pi \times 2}{3}\right)$$

$$= (-0.5 - i\,0.866) \text{ or } 1e^{i\frac{4\pi}{3}}$$

We have:

$$x_0^3 = 1^3 = 1$$

$$x_1^3 = (1e^{i2\pi/3})^3 = 1$$

$$x_2^3 = (1e^{i4\pi/3})^3 = 1$$

In general, we can write

$$\sqrt[n]{z} = (z)^{1/n} = [r.e^{i\theta}]^{1/n}$$

$$= [r.(\cos\theta + i\sin\theta)]^{1/n}$$

$$= r^{1/n}.\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

where $k = 0, 1, 2, 3 \ldots (n-1)$

Homework

(1) Calculate $\sqrt[3]{8}$

(2) Calculate $\sqrt[4]{(3+4i)}$