U3A Maths

18-Jun-2024 Term 2 / Week 8

Complex Numbers - Power & Roots.

- Powers of complex Numbers

$$z^n = (r.e^{i\theta})^n$$

$$= r^n . e^{in\theta}$$

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

De Moivre's Theorem

- Using Euler's equation, the above derivation is simple.
- However, the above equation was derived by De Moivre by Mathematical induction much before Euler's Equation!

- He derived by logical induction:

$$z = r(\cos\theta + i\sin\theta) = (x+iy)$$

$$z^2 = r^2(\cos\theta + i\sin\theta)^2$$

$$= r^2(\underbrace{\cos^2\theta - \sin^2\theta} + 2i \underbrace{\cos\theta.\sin\theta})$$

$$= r^2[\cos(2\theta) + i\sin(2\theta)]$$

- De Moivre extended the above process and obtained the general equation for z^n
- The main feature of De Moivre's theorem is that it can be extended to complex Roots!

- The n^{th} root of a complex number can be obtained using Euler's equation

$$\sqrt[n]{z} = z^{1/n} = (r.e^{i\theta})^{1/n}$$

$$= r^{1/n}(\cos(\theta/n) + i\sin(\theta/n))$$

- However, n^{th} root of a number must have 'n' roots. The application of Euler's equation gives only one root!

For Ex $\sqrt[3]{8} = 2$

how about the other roots?

- The general equation for 'n' roots is as below:

$$(z)^{1/n} = [r(\cos\theta + i\sin\theta)]^{1/n}$$

$$= r^{1/n}(\cos\theta + i\sin\theta)^{1/n}$$

The roots are:

$$z_k = r^{1/n}\left[\cos\left(\frac{\theta+2\pi k}{n}\right) + i\sin\left(\frac{\theta+2\pi k}{n}\right)\right]$$

where $k = 0, 1, 2, \ldots (n-1)$ correspond to 'n' roots

[Note: 2π radians $= 360°$ or 1 sin/cos cycle]

- The above equation shows the power of De Moivre's theorem.

Home Work Problems

① Calculate $\sqrt[3]{8}$

Let $Z = (8 + i0)$ or $8e^{i0^\circ}$

$\therefore$ we have $r = 8$ & $\theta = 0^\circ$

Using De Moivre's Equation

Root 1 ; set $k=0$

$$Z_0 = (r^{1/3})\left[\cos\left(\frac{2\pi \times 0 + 0}{3}\right) + i\sin\left(\frac{2\pi \times 0 + 0}{3}\right)\right]$$

$$= (8)^{1/3}[\cos(0) + i\sin(0)]$$

$$= 2\cdot(1 + i0) = (2 + i0)$$

or 2

Root 2 set $k = 1$

$$Z_1 = (8)^{1/3}\left[\cos\left(\frac{2\pi\times 1 + 0}{3}\right) + i\sin\left(\frac{2\pi \times 1 + 0}{3}\right)\right]$$

$$= 2\left[\cos(2\pi/3) + i\sin\left(\frac{2\pi}{3}\right)\right]$$

$$= 2[-0.5 + i0.866]$$

$$= (-1 + i1.732) = 2e^{i2\pi/3}$$

Check $(Z_1)^3 = \left(2e^{i2\pi/3}\right)^3$

$$= 2^3 \cdot e^{i2\pi}$$

$$= 8(\cos(2\pi) + i\sin(2\pi))$$

$$= 8(1 + i0)$$

$$= 8 \text{ !!}$$

Root 3 set $k = 2$

$$Z_2 = (8)^{1/3}\left[\cos\left(\frac{2\pi\times 2 + 0}{3}\right) + i\sin\left(\frac{2\pi \times 2 + 0}{3}\right)\right]$$

$$= 2[\cos(4\pi/3) + i\sin(4\pi/3)]$$

$$= 2(-0.5 - i0.866)$$

$$= (-1 - i1.732) \text{ or } 2e^{i4\pi/3}$$

② Calculate $\sqrt[4]{(3+4i)}$

Let $Z = (3 + 4i)$ or $5e^{i0.927}$ radians

or $5e^{i53^\circ 13'}$

$\therefore$ we have $r = 5$ $\theta = 0.927$ radians.

$K=0$

$$Z_0 = (5^{1/4})\left[\cos\left(\frac{2\pi\times 0 + 0.927}{4}\right) + i\sin\left(\frac{2\pi\times 0 + 0.927}{4}\right)\right]$$

$$= 1.495(0.973 + i0.23)$$

$$= (1.455 + i0.344) \text{ or } 1.495e^{i0.232}$$

$K=1$

$$Z_1 = (5^{1/4})\left[\cos\left(\frac{2\pi\times 1 + 0.927}{4}\right) + i\sin\left(\frac{2\pi\times 1 + 0.927}{4}\right)\right]$$

$$= 1.495(-0.23 + i0.973) \text{ or } 1.495e^{i1.8}$$

$K=2$

$$Z_2 = (5^{1/4})\left[\cos\left(\frac{2\pi\times 2 + 0.927}{4}\right) + i\sin\left(\frac{2\pi\times 2 + 0.927}{4}\right)\right]$$

$$= 1.495(-0.973 - i0.23) \text{ or } 1.495e^{i3.373}$$

$K=3$

$$Z_3 = (5)^{1/4}\left[\cos\left(\frac{2\pi\times 3 + 0.927}{4}\right) + i\sin\left(\frac{2\pi\times 3 + 0.927}{4}\right)\right]$$

$$= 1.495(0.23 - i0.973) \text{ or } 1.495e^{i4.944}$$

Note: $Z_2 = -Z_0$ & $Z_3 = -Z_1$