U3A Maths

2-Jul-2024 Term 2/Week 10

Complex Numbers & A.C. Circuit Analysis

Review

- D.C. circuit analysis is done using Ohm's law, namely

$$\text{Current}(I) = \frac{\text{Voltage}(V)}{\text{Resistance}(R)}$$

- A.C. source voltage varies sinusoidally, hence current also varies accordingly.
- Let AC source voltage be

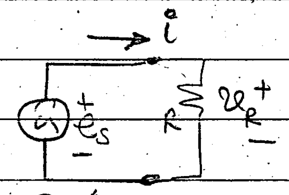
$$e_s = E_{sm}\cos(2\pi f t) \text{ volts}$$

where f = no. of cycles/sec

- For Resistance:

Ohm's law

$$e_s = v_R = iR$$



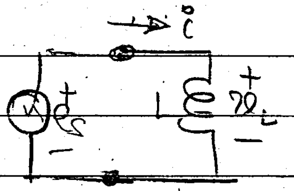
$$\therefore i = \frac{e_s}{R} = \frac{E_{sm}\cos(\omega t)}{R} \text{ Amps}$$

(where $\omega = 2\pi f$)

- for inductance (magnetic field):

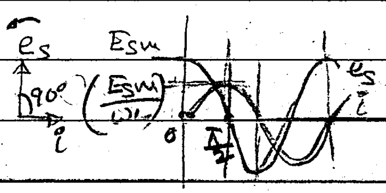
Faraday's law:

$$e_s = v_L = L\cdot\frac{di}{dt}$$



$$\therefore i = \frac{1}{L}\int e_s\,dt = \frac{1}{L}\int E_{sm}\cos(\omega t)dt$$

$$= \left(\frac{E_{sm}}{\omega L}\right)\sin(\omega t)$$

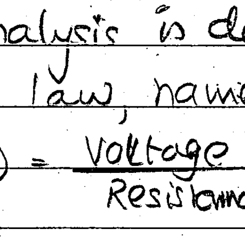


Note: current lags voltage by $\left(\frac{\pi}{2}\right)$ rad.

- R-L Circuit

$$e_s = v_R + v_L$$

$$= iR + L\cdot\frac{di}{dt}$$



This problem becomes difficult to solve! of course, in practice we do have more extensive circuits.

Solution Using Complex Numbers

Let us represent the source voltage as a complex number ($\bar{E}_s$)

$$\bar{E}_s = E_{sm}e^{j\omega t}$$

(Note: we have used 'j' $=\sqrt{-1}$ instead of 'i' $=\sqrt{-1}$)

Using Euler's equation, we have

$$\bar{E}_s = E_{sm}\cos(\omega t) + j\,E_{sm}\sin(\omega t)$$

$\therefore$ source voltage

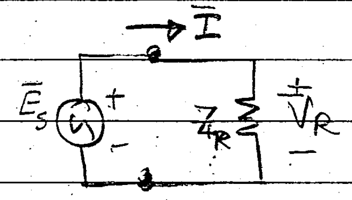
$$e_s = \text{Real Part of }(\bar{E}_s)$$

$$= E_{sm}\cos(\omega t) \text{ volts!}$$

- Let us now see the advantage (power!) of using complex numbers!

Resistance

Since, the source voltage is a complex numer, let us represent other parameters in the circuit as complex numbers.



Hence, we have:

Current$(i) \Rightarrow \bar{I}$

Resistance $(R) \Rightarrow \bar{Z}_R = (R + j0)$

$= Re^{j0}$

Resistor Voltage$(v_R) \Rightarrow \bar{V}_R$

Hence, we have

$$\bar{E}_S = \bar{V}_R = \bar{I}\,\bar{Z}_R$$

$$\therefore \bar{I} = \frac{\bar{E}_S}{\bar{Z}_R} = \frac{E_{sm} \cdot e^{j\omega t}}{R \cdot e^{j0}}$$

$$= \left(\frac{E_{sm}}{R}\right) e^{j\omega t}$$

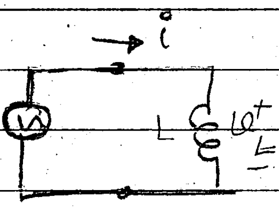
$$\therefore i = \text{Real Part of } \left[\frac{E_{sm}}{R} \cdot e^{j\omega t}\right]$$

However, in complex form we have $\boxed{\bar{I} = \frac{\bar{E}_S}{\bar{Z}_R}}$ ohm's Law for A.C.!

Inductance

Recall that

$e_s = v_L = L \cdot \frac{di}{dt}$



We now have:

$$e_s = \text{Real Part of } (E_{sm}\, e^{j\omega t})$$

$$\therefore i = \frac{1}{L}\int e_s\, dt = \frac{1}{L}\int E_{sm}\, e^{j\omega t}\, dt$$

Note: let us keep "Real Part of" in the back of our mind!

We have, after integration

$$i = \frac{1}{(L)(j\omega)} E_{sm} \cdot e^{j\omega t}$$

$$= \frac{E_{sm}\, e^{j\omega t}}{(j\omega L)} = \frac{E_{sm}\, e^{j\omega t}}{\omega L\, e^{j(\pi/2)}}$$

$$\therefore i = \left(\frac{E_{sm}}{\omega L}\right) e^{j(\omega t - \pi/2)}$$

- We are interested only in the "Real Part"!

$$\therefore i = \frac{E_{sm}}{\omega L} \cos(\omega t - \pi/2)$$

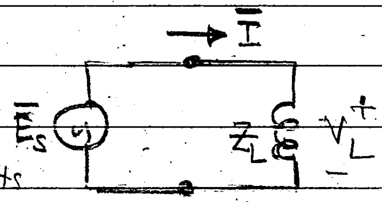
$$i = \frac{E_{sm}}{\omega L} \sin(\omega t) \text{ Amps}$$

Same result as before for Inductance!

- However, it is much simpler to leave the values in complex form by replacing the inductance 'L' as a complex number, namely "$j\omega L$".

ie, Inductance$(L) \Rightarrow Z_L = (0 + j\omega L)$

Using complex values:



$\bar{E}_S = E_{sm}\, e^{j\omega t}$ volts

$\bar{Z}_L = (0 + j\omega L)$

$= \omega L\, e^{j\pi/2}$

$\therefore$ we have

$$\bar{I} = \frac{\bar{E}_S}{Z_L} \text{ or } \frac{E_{sm}\, e^{j\omega t}}{\omega L\, e^{j\pi/2}}$$

This form is more convenient

Also, we have

$$\bar{V}_L = \bar{E}_S$$

$\therefore$ for inductance, we can write

$$\boxed{\bar{I} = \frac{\bar{V}_L}{\bar{Z}_L}}$$

Ohm's law for Inductance in A.C.!

$\therefore$ we can solve AC circuits similar to DC!!