U3A Maths

30-JUL-2024 Term 3 / Week 2

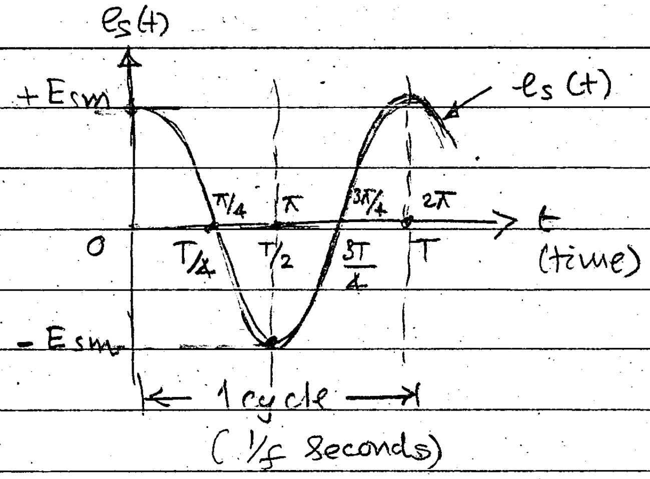
Complex Numbers in A.C. Circuit Analysis

A.C. Source

- An A.C. Source voltage (e_s) varies sinusoidally w.r.t. time.
- Let us say that the number of sinusoidal cycles / second is 'f'
- We can represent the sinusoidal variations mathematically as below:

$$e_s = e_s(t) = E_{sm} \cdot \cos(2\pi f t) \quad \text{volts}$$

[In practice, 'e_s' is treated as identical to '$e_s(t)$']



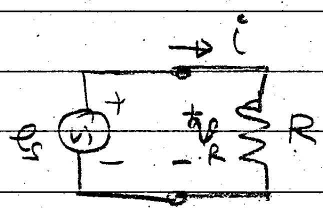
- time for one cycle (T) = $1/f$ sec.

For ex: given $f = 50$ Hz & $E_{sm} = 100$ V the values of voltage at time $t = 10$ ms or 10×10^{-3} second

$$e_s(10\,\text{ms}) = E_{sm} \cos(2\pi \times 50 \times 10 \times 10^{-3})$$
$$= 100 \cos(\pi)$$
$$= -100\,\text{V}$$

A.C. source connected to Resistance

- Note that lower case variables represent time varying values.



- Also, "$2\pi f$" is denoted by "ω"

∴ we have:

$$e_s = E_{sm} \cos(\omega t) \quad \text{volts}$$

$$v_R = i R \quad \text{– ohm's law}$$

$$e_s - v_R = 0 \quad \text{– Kirchoff's Voltage Law (K.V.L.)}$$

$$\therefore \; e_s = v_R = i R$$

or $$i = \frac{e_s}{R} = \frac{E_{sm} \cos(\omega t)}{R}$$

$$\therefore \; i = \left(\frac{E_{sm}}{R}\right) \cos(\omega t) = I_m \cos(\omega t)$$

where, 'I_m' is the maximum value of the current

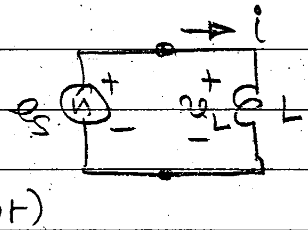
$$\therefore \; I_m = \left(\frac{E_{sm}}{R}\right)$$

- the current 'i' follow "Cosine" wave in phase with 'e_s'.
- Hence, the solution of AC circuits with Resistances is straight forward.
- We do not need complex numbers!

A.C. source connected to Inductance

Given 'e_s' & 'L'

Solve for 'i'



We have:

$$e_s = E_{sm} \cos(\omega t)$$

$$v_L = L \cdot \frac{di}{dt} \quad \text{– Faraday's law}$$

Also $e_s - v_L = 0$ — K.V.L.

we have

$$e_s = v_L = L.\frac{di}{dt}$$

$\therefore\ i = \frac{1}{L}\int e_s . dt = \frac{1}{L}\int E_{sm} \cos(\omega t) dt$

Integration becomes much easier if we express

$$e_s = \text{Real Part of } [E_{sm} e^{j\omega t}]$$

$$E_{sm} e^{j\omega t} = \underbrace{E_{sm}\cos(\omega t)}_{\text{Real Part}} + j\, E_{sm}\sin(\omega t)$$

$\therefore\ i = \frac{1}{L}\int E_{sm} e^{j\omega t}\ dt$

Note: we can use only the "Real part" after integration!

$$i = \frac{E_{sm}}{j\omega L} . e^{j\omega t}$$

Note: $\left[\int e^{ax} dx = \frac{1}{a}.e^{ax}\right]$

$\therefore$ we have

$$i = \frac{E_{sm} e^{j\omega t}}{j\omega L} \quad (\text{let } X_L = \omega L)$$

$$\boxed{i = \frac{E_{sm} . e^{j\omega t}}{X_L\, e^{j90^\circ}}} \text{ — } \underline{\text{Eqn (1)}}$$

Hence $i = \underset{\text{part}}{\text{Real}}\left[\frac{E_{sm}}{X_L} . e^{j(\omega t - 90^\circ)}\right]$

ie, $i = \frac{E_{sm}}{X_L} \cos(\omega t - 90^\circ)$ at time 't'

- The current (i) through the inductance is a "cosine" wave with a phase lag of 90° to voltage (e_s)

- Since, both 'e_s' & 'i' are "cosine" waves, they can be constructed by knowing only the values at t=0!

At time t=0, Eqn (1) becomes

$$i = \frac{E_{sm}\, e^{j0}}{X_L\, e^{j90^\circ}}$$

By treating the above equation complex numbers, we have

$$\bar{I} = \frac{\bar{E}_s}{\bar{Z}_L} \Rightarrow \text{Ohm's law (in complex form)!}$$

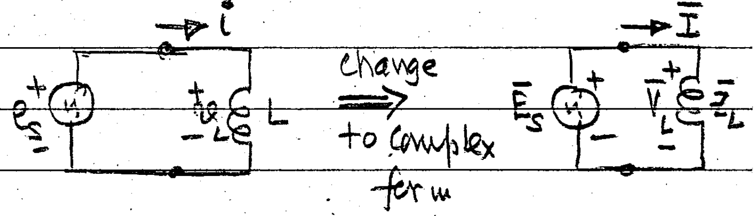
where $\bar{E}_s = E_{sm} e^{j0}$ & $\bar{Z}_L = X_L e^{j90^\circ}$

$= (E_{sm} + j0) \qquad = (0 + jX_L)$

- Hence, our problem is now transformed from a Differential equation to an Algebraic Equation (but with complex numbers!)
- This makes solution of general AC circuits much simpler.

Ex: Given $e_s = 100\cos(\omega t)$ & $f = 50$ Hz and $L = 10$ mH

Calculate the current flow (i)



We have: $\bar{E}_s = E_{sm} e^{j0^\circ} = 100 e^{j0^\circ}$

or in Polar form $\Rightarrow = 100\angle 0^\circ$ V

$\bar{Z}_L = X_L e^{j90^\circ} = (\omega L) e^{j90^\circ}$

$= (2\pi f L)\, e^{j90^\circ}$

$= (2\pi \times 50 \times 10 \times 10^{-3})\, e^{j90^\circ}$

$= 3.1416\, e^{j90^\circ} = 3.1416\angle 90^\circ$

$\therefore\ \bar{I} = \frac{100\angle 0^\circ}{3.1416\angle 90^\circ} = \underline{31.83\angle -90^\circ \text{ A}}$

At t=0: i = Real Part $[\bar{I}] = 31.83\cos(-90^\circ)$

At any 't' $\quad \underline{i = 31.83\cos(\omega t - 90^\circ)}$ Amps