U3A Maths

20-Jul-2024 Term 3 / Week5

Fractional Roots

Review

Complex Number — From Euler's Theorem

$$Ae^{i\theta} = A(\cos\theta + i\sin\theta)$$
$$= A\,cis(\theta) \leftarrow \text{Simplified notation}$$

- Note that $cis(\theta) \equiv \cos(\theta) + i\sin(\theta)$

we are using this notation for our convenience!

- Note also

$$Ae^{i\theta} = \underbrace{A\cos(\theta)}_{\text{Real part}} + i\underbrace{A\sin(\theta)}_{\text{Imaginary Part}}$$
$$= (A_x + iA_y)$$

iy, A, θ, A_y, A_x, x

De Moivre's Theorem for Roots

For n^{th} root of complex no.

$$\sqrt[n]{Ae^{i\theta}} = (Ae^{i\theta})^{1/n}$$
$$= A\,cis\left(\frac{2\pi k + \theta}{n}\right)$$

for $k = 0, 1, 2, \ldots (n-1)$

For real numbers

$$\sqrt[n]{Ae^{i0}} = (Ae^{i0})^{1/n} \qquad \text{Note that } \theta = 0$$
$$= (A)^{1/n}\,cis\left(\frac{2\pi k + 0}{n}\right)$$
$$= \sqrt[n]{A}\,cis\left(\frac{2\pi k}{n}\right) \text{ or } \sqrt[n]{A}\,cis\left(\frac{360^\circ k}{n}\right)$$

for $k = 0, 1, 2, \cdots (n-1)$

Ex:

Calculate roots of $\sqrt[2.5]{2}$

$$\sqrt[2.5]{2} = (2)^{1/2.5} = (2)^{1/(5/2)} = 2^{2/5}$$

We have $2^{2/5} = (2^2)^{1/5} = \sqrt[5]{4}$

ie, 5th root of 4!

Using De Moivre's theorem, we have $n = 5$ & $k = 0, 1, 2, 3 \& 4$

For $k = 0$

$$R_0 = \sqrt[5]{4}\,cis\left(\frac{360^\circ \times 0}{5}\right)$$
$$= 1.3195\,cis(0^\circ)$$
$$= 1.3195(\cos 0^\circ + i\sin 0^\circ) = 1.3195$$

$k = 1$

$$R_1 = \sqrt[5]{4}\,cis\left(\frac{360^\circ \times 1}{5}\right)$$
$$= 1.3195\,cis(72^\circ)$$

For simplicity, leave the result as "cis"

$k = 2$

$$R_2 = \sqrt[5]{4}\,cis\left(\frac{360^\circ \times 2}{5}\right)$$
$$= 1.3195\,cis(144^\circ)$$
$$= 1.3195\,cis(2 \times 72^\circ)$$

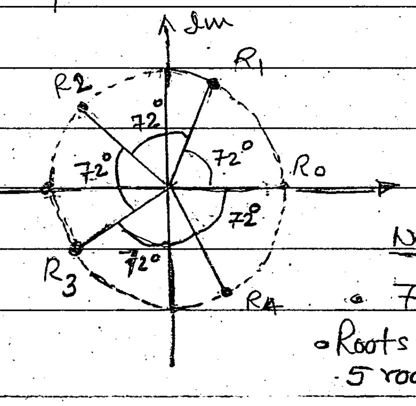
For $k = 3$

$$R_3 = \sqrt[5]{4}\,cis\left(\frac{360^\circ \times 3}{5}\right)$$
$$= 1.3195\,cis(216^\circ)$$
$$= 1.3195\,cis(3 \times 72^\circ)$$

For $k = 4$

$$R_4 = \sqrt[5]{4}\,cis\left(\frac{360^\circ \times 4}{5}\right)$$
$$= 1.3195\,cis(288^\circ)$$
$$= 1.3195\,cis(4 \times 72^\circ)$$

We can represent the above roots graphically



Notes

- $72^\circ \times 5 = 360^\circ$
- Roots repeat after 5 roots!

- We have found the 5 roots, presumably of $\sqrt[2.5]{2}$.
- Proof the pudding is in eating it!
- Let's check our results.

For Ex. $\sqrt{4} = +2$ and -2

we have $(+2)^2 = (+4)$ ✓ & $(-2)^2 = (+4)$ ✓

- We must get back the original value for 2.5^{th} Power of the roots.

$$(R_0)^{2.5} = (1.3195)^{2.5} = 2 \checkmark$$

$$(R_1)^{2.5} = [1.3195 \text{ cis}(72°)]^{2.5}$$

- Note that

$$(\cos\theta + i\sin\theta)^n = [\cos(n\theta) + i\sin(n\theta)]$$

De Moivre's theorem.

$$\therefore R_1 = (1.3195)^{2.5}(\cos 72° + i\sin 72°)^{2.5}$$
$$= 2 \cdot [\cos(2.5\times72°) + i\sin(2.5\times72°)]$$
$$= 2[\cos(180°) + i\sin(180°)]$$
$$= 2[-1 + i0]$$
$$= \underline{-2}$$ ← This is not right we must have +2!

Similarly

$$R_2 = (1.3195)^{2.5}[\cos(144°) + i\sin(144°)]^{2.5}$$
$$= 2(1 + i0) = \underline{+2} \checkmark$$

$$R_3 = (1.3195)^{2.5}[\cos(216°) + i\sin(216°)]^{2.5}$$
$$= 2(-1 + i0)$$
$$= \underline{-2}$$ ⇐ Incorrect value! we should have +2

$$R_4 = (1.3195)^{2.5}[\cos(288°) + i\sin(288°)]^{2.5}$$
$$= 2(1 + i0) = \underline{+2} \checkmark$$

- Hence,

Roots R_0, R_2 & R_4 are correct!

Roots R_1 & R_3 are incorrect!

$\therefore$ we have only 3 roots!!

- Lastly, we actually calculated $\sqrt[5]{4}$ which must have 5 roots

{ • Check that $\sqrt[5]{4}$ has 5 valid roots. R_0, R_1, R_2, R_3 & R_4 are valid roots! } Home Work!

In conclusion, refering to "USA Notes" dated 6-Jul-2024;

- **Phil's method** is same as calculated above. However, we have to check for validity of the Roots!
- **Stewart's method** uses $n = 2.5$ and calculate $\sqrt[2.5]{2}$

$$R_K = (2)^{1/2.5} \text{ cis}\left(\frac{360\times K}{2.5}\right)$$

for $K = 0, 1, 2, 3$ & 4

- It is interesting to note that, in Stewart's method, the first three roots (R_0, R_1 & R_2) are valid roots and the last two roots (R_3 & R_4) are invalid!
- Check above! **[Home Work]**!!