U3A Maths

27-Aug-2024 Term 3 / Week 6

Fractional Roots

Review

- Complex number provides for general form of numbers.
- It includes representation of square root of negative numbers
- $A e^{i\theta} = \underbrace{A(\cos\theta + i\sin\theta)}_{\text{Euler's theorem}} \equiv \underbrace{A\,cis(\theta)}_{\text{Simplified Notation.}}$

Ex: 1 $\quad 3 \Rightarrow 3e^{i0^\circ} = 3(\cos 0^\circ + i\sin 0^\circ)$

$= \underline{3}$

Ex: 2 $\quad \sqrt{-4} \Rightarrow \sqrt{4}\cdot\sqrt{-1} = 2i$

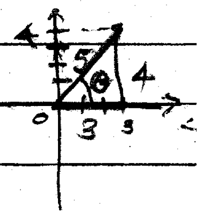
$= 2e^{i90^\circ} = 2(\cos 90^\circ + i\sin 90^\circ)$

$= 2i$

Ex: 3 $\quad 3 + \sqrt{-16}$

$= 3 + \sqrt{16}\cdot\sqrt{-1} = (3+4i)$

$= \sqrt{3^2+4^2}\; e^{i(\tan^{-1}\frac{4}{3})}$



$(3+\sqrt{-16}) = 5e^{i\,53.13^\circ}$

$= A e^{i\theta}$ where $A = 5$

& $\theta = 53.13^\circ$

Ex: 4 $\quad 5e^{i\,36.87^\circ}$

$= 5(\cos 36.87^\circ + i\sin 36.87^\circ)$

$= 5(0.8 + i\,0.6)$

$= (4 + i3)$

$= 4 + \sqrt{-1}\,.\,3$

$= 4 + \sqrt{-1}\,\sqrt{9}$

$\underline{5e^{i\,36.87^\circ}} = \underline{(4 + \sqrt{-9})}$

- Powers of complex numbers

$(A e^{i\theta})^n = A^n e^{in\theta}$

$= A^n(\cos n\theta + i\sin n\theta)$

$= A^n\, cis(n\theta)$

Also called De Moivre's theorem

- Roots of complex number

$(A e^{i\theta})^{1/n}$ where 'n' can be a non-integer

The Roots are

$$R_K = (A^{1/n})\left[\cos\left(\frac{2\pi K + \theta}{n}\right) + i\sin\left(\frac{2\pi K + \theta}{n}\right)\right]$$

for $K = 0, 1, 2, \cdots (n-1)$

Note: For a real number 'A', $\theta = 0^\circ$

$A e^{i\theta} \Rightarrow A e^{i0} = A$

Home work

Find the valid roots of $\sqrt[3.5]{4}$

Phil's Method

$(4)^{1/3.5} = (4)^{1/(7/2)} = 4^{2/7}$

$= (4^2)^{1/7} = (16)^{1/7}$

$\therefore\ (16^{\,i0^\circ})^{1/7}$ we have $\theta = 0^\circ$

$n = 7$

$\therefore\ R_K = (16)^{1/7}\cdot cis\left(\frac{2\pi K}{7}\right)$

where $K = 0, 1, 2, 3, 4, 5, 6$

$$\therefore\ R_0 = (16)^{1/7}\, cis\left(\frac{2\pi\times 0}{7}\right)$$
$$= 16^{1/7}\, cis(0^\circ) = \underline{\underline{1.486\angle 0^\circ}}$$

$$R_1 = (16)^{1/7}\, cis\left(\frac{2\pi\times 1}{7}\right) = \underline{\underline{1.486\angle 51.43^\circ}}$$

$$R_2 = (16)^{1/7}\, cis\left(\frac{2\pi\times 2}{7}\right) = 1.486\angle 102.86^\circ$$

$$R_3 = (16)^{1/7}\, cis\left(\frac{2\pi\times 3}{7}\right) = 1.486\angle 154.29^\circ$$

$$R_4 = (16)^{1/7}\, cis\left(\frac{2\pi\times 4}{7}\right) = 1.486\angle 205.71^\circ$$

$$R_5 = (16)^{1/7}\, cis\left(\frac{2\pi\times 5}{7}\right) = 1.486\angle 257.14^\circ$$

$$R_6 = (16)^{1/7}\, cis\left(\frac{2\pi\times 6}{7}\right) = 1.486\angle 308.57^\circ$$

The above are supposed to be 3.5^{th} Roots of 4! Let's check!!

<u>Verify validity of Roots for $\sqrt[3.5]{4}$</u>

$$(R_0)^{3.5} = (1.486\angle 0^\circ)^{3.5} = \underline{\underline{4}}\ \checkmark$$

$$(R_1)^{3.5} = (1.486\angle 51.43^\circ)^{3.5}$$
$$= (1.486)^{3.5}\, cis(3.5\times 51.43^\circ)$$
$$= 4\left[\cos(3.5\times 51.43^\circ) + i\sin(3.5\times 51.43^\circ)\right]$$
$$= 4(-1+i0) = \underline{\underline{-4}}\ \times$$

$$(R_2)^{3.5} = (1.486)^{3.5}\, cis(3.5\times 102.86^\circ)$$
$$= 4(1+i0) = \underline{\underline{4}}\ \checkmark$$

$$(R_3)^{3.5} = (1.486)^{3.5}\, cis(3.5\times 154.29^\circ)$$
$$= 4(-1+i0) = -4\ \times$$

$$(R_4)^{3.5} = (1.486)^{3.5}\, cis(3.5\times 205.71^\circ)$$
$$= 4(1+i0) = \underline{\underline{4}}\ \checkmark$$

$$(R_5)^{3.5} = (1.486)^{3.5}\, cis(3.5\times 257.14^\circ)$$
$$= 4(-1+i0) = \underline{\underline{-4}}\ \times$$

$$(R_6)^{3.5} = (1.486)^{3.5}\, cis(3.5\times 308.57^\circ)$$
$$= 4(1+i0) = \underline{\underline{4}}\ \checkmark$$

Hence, <u>R_0, R_2, R_4 & R_6</u> are valid roots

<u>Stewart's Method</u>

$$(4)^{1/3.5} = (4e^{i0})^{1/3.5}$$

$$\therefore\ \theta = 0^\circ \ \ \&\ \ n = 3.5$$

$$R_K = (4)^{1/3.5}\, cis\left(\frac{2\pi K}{3.5}\right)$$

for $K = 0, 1, 2, 3$ (i.e, $K \leq 3.5$)

$$\therefore\ R_0 = (4)^{1/3.5}\, cis(0^\circ)$$
$$= \underline{\underline{1.486\angle 0^\circ}}\ \checkmark$$

$$R_1 = (4)^{1/3.5}\, cis\left(\frac{2\pi\times 1}{3.5}\right)$$
$$= \underline{\underline{1.486\ cis(102.86^\circ)}}\ \checkmark$$

$$R_2 = (4)^{1/3.5}\, cis\left(\frac{2\pi\times 2}{3.5}\right)$$
$$= \underline{\underline{1.486\ cis(205.71^\circ)}}\ \checkmark$$

$$R_3 = (4)^{1/3.5}\, cis\left(\frac{2\pi\times 3}{3.5}\right)$$
$$= \underline{\underline{1.486\ cis(308.57^\circ)}}\ \checkmark$$

We get the <u>valid roots</u> directly!!