

24-Sep-2024

Term 3 / Week 10

Napier Mirifici Logarithmorum

(1614)

- Napier logarithm can be written as

$$y = (0.9999999)^x \cdot 10^7$$

$$= (1 - 10^{-7})^x \cdot 10^7$$

$$\text{or } x = \log_{(1-10^{-7})} (y/10^7)$$

- However, Napier calculated the log table for $x=0,1,2,\dots$ using iterative equation. (with $y_0=10^7$)

$$y_{n+1} = y_n \times (1 - 10^{-7})$$

which requires only subtraction!

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- Why did he use "Sine" values?

- In the 16th Century (ie, 1500's) trigonometric relations were used for navigation based on celestial objects!

- An algorithm called "Prosthaphaeresis" (Greek word!) was used for multiplication and division using formulas from trigonometry.

- "Prosthen" \Rightarrow Before
- "Aphaeresis" \Rightarrow Taking Away (Subtraction)

- The above was extensively used before Napier logarithm for multiplication & division.

- He also associated his log table with "Sin(θ)" values.

Napier Table

<u>x</u>	<u>y</u>	<u>Angle</u>
(logarithm)	Value (sin(θ))	(θ)
0	10,000,000	90° 0'
1	9,999,999	89° 59'
2	9,999,998	89° 58"
3	9,999,997	(Ignored!)
4	9,999,996	89° 57'
...	...	(Ignored)
7	9,999,993	89° 56'
	}	

- Note that 'y' values correspond sin(θ) values!

$$\sin(89^\circ 56') = 0.9999993$$

as per calculator!

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- "Prosthaphaeresis" included trigonometric equations such as:

$$\sin(A) \sin(B) = \frac{\cos(A-B) - \cos(A+B)}{2}$$

- Ex:

Multiply 309×78.8 using prosthaphaeresis!

$$309 = 0.309 \times 10^3 \Rightarrow \sin^{-1}(0.309) \cong 18^\circ \Rightarrow 'a'$$

$$78.8 = 0.788 \times 10^2 \Rightarrow \sin^{-1}(0.788) \cong 52^\circ \Rightarrow 'b'$$

Using the above equ:

$$\sin(18^\circ) \times \sin(52^\circ) = \frac{\cos(18^\circ - 52^\circ) - \cos(18^\circ + 52^\circ)}{2}$$

$$= \frac{\cos(-34^\circ) - \cos(70^\circ)}{2}$$

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$$= \frac{0.82904 - 0.34202}{2}$$

$$= \underline{\underline{0.24351}}$$

∴ We have $0.309 \times 0.788 = 0.24351$

$$\text{Hence, } 309 \times 78.8 = 0.309 \times 10^3 \times 0.788 \times 10^2$$

$$= 0.24351 \times 10^5$$

$$= \underline{\underline{24351}}$$

(Very close to actual result of 24349.2)
 (Note: we used approx values for \sin^{-1})

• During 1500's 'sin' and 'cos' tables were available with 8 to 10 digit precision!

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• Referring to last row of last 2 columns:

$$\sin(89^\circ 30') = \sin(89.5)$$

$$= \underline{\underline{0.9999619}}$$

(Note: table value $\div 10^7$)

• Referring to last row of first 2 columns

$$\sin(0^\circ 30') = \sin(0.5)$$

$$= \underline{\underline{0.0087265}}$$

• Also, note that

$$\cos(89.5) = \sin(0.5) = 0.0087265$$

$$\cos(0.5) = \sin(89.5) = 0.9999619$$

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• Napier's log table enabled multiplication & division without recourse to 'sin' & 'cos'.

• However, provision of 'sin' & 'cos' values made the transition to log table more convenient?!

• Page 1 of Napier's table is included along with this notes.

• Note that the table cleverly provides 'sin' & 'cos' values.

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• Hence, the table provides 'sin' and 'cos' values and their logarithms!

• Finally, what does the middle column (difference of log values) provide?

• For our example, the difference is

$$\log(\sin(89.5)) - \log(\cos(89.5))$$

$$= \log\left(\frac{\sin(89.5)}{\cos(89.5)}\right) = \log(\tan(89.5))$$

Hence, the table provides log of 'tan' values!!

Napier Logarithm - First page

0 Deg	Min	Sinus	Logarithmi	Differentie	Logarithmi	Sinus	Min
0° 0'	A	sin(A) or cos(B)	log (sin(A) or cos(B))	log(sin(A)) - log(cos(A))	log (sin(B) or cos(A))	sin(B) or cos(A)	B
	0	0	Infinitum	Infinitum	0	10,000,000	60
	1	2,909	81,425,681	81,425,680	1	9,999,999	59
	2	5,818	74,494,213	74,494,211	2	9,999,998	58
	3	8,727	70,439,564	70,439,560	4	9,999,996	57
	4	11,636	67,562,746	67,562,739	7	9,999,993	56
	5	14,544	65,331,315	65,331,304	11	9,999,989	55
	6	17,453	63,508,099	63,508,083	16	9,999,984	54
	7	20,362	61,966,595	61,966,573	22	9,999,978	53
	8	23,271	60,631,284	60,631,256	28	9,999,972	52
	9	26,180	59,453,453	59,453,418	35	9,999,965	51
	10	29,089	58,399,857	58,399,814	43	9,999,957	50
	11	31,998	57,446,759	57,446,707	52	9,999,948	49
	12	34,907	56,576,646	56,576,584	62	9,999,938	48
	13	37,815	55,776,222	55,776,149	73	9,999,927	47
	14	40,724	55,035,148	55,035,064	84	9,999,916	46
	15	43,633	54,345,225	54,345,129	96	9,999,904	45
	16	46,542	53,699,843	53,699,734	109	9,999,891	44
	17	49,451	53,093,600	53,093,477	123	9,999,877	43
	18	52,360	52,522,019	52,521,881	138	9,999,862	42
	19	55,269	51,981,356	51,981,202	154	9,999,846	41
	20	58,177	51,468,431	51,468,261	170	9,999,830	40
	21	61,086	50,980,537	50,980,350	187	9,999,813	39
	22	63,995	50,515,342	50,515,137	205	9,999,795	38
	23	66,904	50,070,827	50,070,603	224	9,999,776	37
	24	69,813	49,645,239	49,644,995	244	9,999,756	36
	25	72,722	49,237,030	49,236,765	265	9,999,735	35
	26	75,630	48,844,826	48,844,539	287	9,999,713	34
	27	78,539	48,467,431	48,467,122	309	9,999,691	33
	28	81,448	48,103,763	48,103,431	332	9,999,668	32
	29	84,357	47,752,859	47,752,503	356	9,999,644	31
	30	87,265	47,413,852	47,413,471	381	9,999,619	30

89° 30'

89 Deg

Notes: (1) Sinus = (0.9999999^A * Logarithmi) x 10⁷ (2) 'Sinus' ('sine') = sine function value x 10⁷