U3A Maths

17-Sep-2024 Term 3 / Week 9

Logarithms – John Napier

- If $y = a^x$
  then we define the "logarithmic" function as

$$x = \log_a(y)$$

where 'x' is the logarithm of 'y' for base 'a'.

- There is nothing holy about the "base" value, but it is normally chosen as per

-2-

(calculating)

the convenience of establishing the logarithmic (Log) table

- Last week, we saw that Joost Burgi chose a base value of "1.0001", which provided for simpler calculation to establish (log) table.

- Burgi's log table was versatile and was used for practical calculations, involving multiplication, division, power (y^n) and root $\sqrt[n]{y}$ (where 'n' can be a fraction!)

-3-

- Even though Burgi's table was logical & straight forward, it did not survive the test of time.
- The log table produced by John Napier (1614) become popular and was developed further into more useful forms.

- In fact, the term "logarithm" was introduced by Napier, in his book,

"Mirifici Logarithmorum canonis descriptio" (1614 AD)

"Mirifici" ⇒ "Marvelous" (French origin)

-4-

- Without his knowledge, Napier used a base of 0.9999999 !!
  We will see how he arrived at this base value!
- Similar to Burgi, Napier used a multiplication factor of 10^7.
  $10^7 = 10,000,000 = 10$ Million
- Hence, we have

$$y = (0.9999999^x) \times 10^7$$

or

$$x = \log_{0.9999999}(y \times 10^{-7})$$

- Let us calculate the Napier's log table, using a modern calculator!

-5-

x	$y = (0.999,999,9)^x \times 10^7$
0	10,000,000 . 000,000,0
1	9,999,999 . 000,000,0
2	9,999,998 . 000,000,1
3	9,999,997 . 000,000,3
4	9,999,996 . 000,000,6
5	9,999,995 . 000,001,0
6	9,999,994 . 000,001,5
7	9,999,993 . 000,002,1
...	...
101	9,999,891 . 000,588,5
205	9,999,795 . 002,090,1
...	...
1,000	9,999,000 . 049,948,3
10,000	9,990,004 . 997,834,2
100,000	9,990,498 . 332,541,4
200,000	9,801,986 . 723,265,6

-6-

- Napier's table is more powerful! We have fairly continuous value of 'y' and the maximum value is limited to 10^7!
- The main question is how did Napier calculate 0.9999999^x? (Note that $0.999,999,9 = (1 - 1/10^7)$)
- He used a very clever method! We have

$$x=0 \Rightarrow y_0 = 10^7$$

$$x=1 \Rightarrow y_1 = y_0\left(1 - \frac{1}{10^7}\right) = y_0 - y_0/10^7 = (10^7 - 1)$$

$$x=2 \Rightarrow y_2 = y_1(1 - 1/10^7) = (y_1 - y_1/10^7)$$

$$\text{Also, } y_2 = y_0(1 - 1/10^7)(1 - 1/10^7) = y_0(1 - 1/10^7)^2$$

$$\therefore x = n+1 \Rightarrow y_{n+1} = (y_n - y_n/10^7)$$

-7-

- Hence, the log table can be obtained by Subtraction!

x	y
0	10,000,000 . 000,000,0 (y_0)
	− 1 . 000,000,0 ($y_0/10^7$)
1	9,999,999 . 000,000,0 (y_1)
	− 0 . 999,999,9 ($y_1/10^7$)
2	9,999,998 . 000,000,1 (y_2)
	− 0 . 999,999,8 ($y_2/10^7$)
3	9,999,997 . 000,000,3 (y_3)
	− 0 . 999,999,7 ($y_3/10^7$)
4	9,999,996 . 000,000,6 y_4

↓

So on

-8-

- The above system works, however, Napier also connected up above table with $\sin(\theta)$ values!
- He used the available Sine table and produced the following table.

Logarithm x	Value (sinus) y ($\sin\theta \times 10^7$)	Angle θ
0	10,000,000	90° 0'
1	9,999,999	89° 59'
2	9,999,998	89° 58'
4	9,999,996	89° 57'
7	9,999,993	89° 56'

- Napier's table provides the log values at every $(1/60)^\circ$ for 0 to 90°!
- He was not interested in other values?!