U3A Maths

15-Oct-2024 — Term 4 / Week 1

Logarithms – Napier Methodology

- Logarithm is formally defined as below:

Given $a^x = y$

then $x = \log_a(y)$

where 'x' is the logarithm of 'y' for base 'a'

For Ex: Given $2^3 = 8$

then $\log_2(8) = 3$

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- The main advantage of logarithms is that they convert multiplication / division to addition / subtraction respectively.

- Even though Napier introduced the term "Logarithm", however, he did not use the above definition to arrive at his log table!

- Without his knowledge, he used
  $$y = (0.999,999,9)^x \times 10^7$$
  to calculate his log table.

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- Using a modern calculator, we can establish Napier's log table as below:

x	$y = (0.999,999,9)^x \times 10^7$
0	10,000,000.000,000,0
1	9,999,999.000,000,0
2	9,999,998.000,000,1
3	9,999,997.000,000,3
4	9,999,996.000,000,6
5	9,999,995.000,001,0
6	9,999,994.000,001,5
...	....

- Notes: The above values can also be calculated using the Eqn:
  $$y_{n+1} = \left(y_n - y_n/10^7\right)$$
  for $n = 1, 2, 3, \ldots$ with $y_0 = 10^7$

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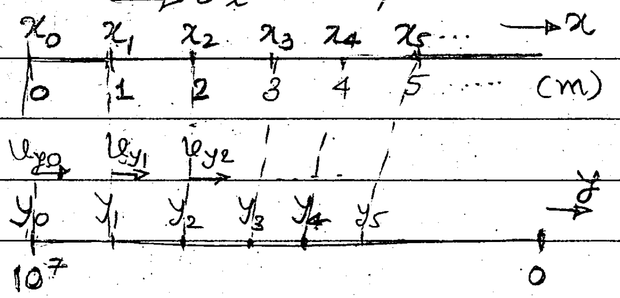
- Napier had an entirely different thought process (philosophy!) to establish his log table!

- Napier imagined two particles travelling along two parallel lines.
  First line ⇒ Infinite Length
  Second line ⇒ Fixed length of 10^7 units

- The first particle moves at a constant velocity, so it travels equal distance for a set time period
- The second particle moves at a velocity proportional to the remaining distance, using same starting point and time period.

- Let us say that the starting velocity is 10^7 m/s and the fixed time period (T) is 10^{-7} s.
  Hence, the first particle moves 1 m for each time period

$\rightarrow v_x = 10^7$ m/s & $T = 10^{-7}$ s



- We have $x_0 = 0$ & $y_0 = 10^7$
  Velocity $v_{x0} = 10^7$ m/s

$$\therefore x_1 = x_0 + v_{x0} \cdot T$$
$$= 0 + 10^7 \times 10^{-7}$$
$$= 1$$

Velocity $v_{y0} = 10^7$ m/s

$$\therefore y_1 = y_0 - v_{y0} \cdot T$$
$$= 10^7 - 10^7 \times 10^{-7}$$
$$= (10^7 - 1)$$

Note that corresponding 'y' values are decreasing

- We now have $x_1 = 1$, $y_1 = (10^7 - 1)$
  Velocity $v_{x1} = 10^7$ m/s

$$\therefore x_2 = x_1 + v_{x1} T$$
$$= 1 + 10^7 \times 10^{-7}$$
$$= 1 + 1 = 2$$

Velocity $v_{y1} = (10^7 - 1)$ m/s

$$\therefore y_2 = y_1 - v_{y1} \cdot T$$
$$= (10^7 - 1) - (10^7 - 1) \times 10^{-7}$$

$$= (10^7 - 1) - (1 - 1 \times 10^{-7})$$
$$\therefore y_2 = (10^7 - 2 + 1 \times 10^{-7})$$

- We now have $x_2 = 2$, $y_2 = (10^7 - 2 + 1 \times 10^{-7})$
  Velocity $v_{x2} = 10^7$ m/s

$$x_3 = x_2 + 10^7 \times 10^{-7} = 2 + 1 = \underline{3}$$

Velocity $v_{y2} = (10^7 - 2 + 1 \times 10^{-7})$

$$\therefore y_3 = y_2 - v_{y2} \cdot T$$
$$= (10^7 - 2 + 1 \times 10^{-7}) - (10^7 - 2 + 1 \times 10^{-7}) \times 10^{-7}$$
$$= (10^7 - 3 + 3 \times 10^{-7} - \cancel{1 \times 10^{-14}})$$
$$= (10^7 - 3 + 3 \times 10^{-7})$$

(Neglecting 10^{-14} terms!)

- Continuing the process we get:

$x_4 = 4$; $y_4 = (10^7 - 4 + 6 \times 10^{-7})$

$x_5 = 5$; $y_5 = (10^7 - 5 + 10 \times 10^{-7})$

Finally we have,

x	y
0	10^7 [10,000,000.000,000,0]
1	$(10^7 - 1)$ [9,999,999.000,000,0]
2	$(10^7 - 2 + 1 \times 10^{-7})$ [9,999,998.000,000,1]
3	$(10^7 - 3 + 3 \times 10^{-7})$ [9,999,997.000,000,3]
4	$(10^7 - 4 + 6 \times 10^{-7})$ [9,999,996.000,000,6]
5	$(10^7 - 5 + 10 \times 10^{-7})$ [9,999,995.000,001,0]

Home work

Guess the 'y' value for $x = 6$ and check with calculation!