U3A Maths

17-Dec-2024 Term 4 / Week 10

Logarithms & Fractional Roots

Review

- We calculated $\sqrt[3]{27}$ and $\sqrt[2]{4}$ using logarithms
- However, the above calculation provides only one root.
- We need the concept or knowledge of complex numbers to calculate other roots.

- The general form of a complex number is provided by Euler's equation.

$$A.\,e^{i\theta} = A(\cos\theta + i\sin\theta)$$

- Any given (real) number can be expressed as a complex number by setting the angle $\theta = 0$ deg or radians.
- For Ex

$$27 \Rightarrow 27\,e^{i0^\circ}$$
$$4 \Rightarrow 4\,e^{i0^\circ}$$

- Note that if $\theta \neq 0^\circ$, then a given number becomes a two-dimensional number or what we call a complex number!
- Euler's form of numbers is the general form of numbers, since they can represent square root of negative numbers & negative numbers.
- Euler's form number accomplishes this by using the variable 'i' which is set as equal to $\sqrt{-1}$ or $i^2 = -1$

- Use of 'i' or $\sqrt{-1}$ enables representation of square root of any given negative number!

For Ex $\sqrt{-4} = \sqrt{(-1)(4)}$
$= \sqrt{(-1)} \cdot \sqrt{4}$
$= i4$ or $4i$

- This makes the number system a "Closed" set.
- De Moivre's theorem:

Given

$$\sqrt[n]{A e^{i\theta}} = \left(A e^{i\theta}\right)^{1/n}$$

The roots are

$$R_K = (A)^{1/n} \cdot e^{i\left(\frac{2\pi K + \theta}{n}\right)}$$

where $K = 0, 1, 2, \ldots, (n-1)$

Note

For fractional roots $K = 0, 1, 2, \ldots (K < n)$

Ex: Find roots of

$$\sqrt[2.5]{4}$$

Using De Moivre's theorem

$$R_K = (4)^{1/2.5} e^{i\left(\frac{2\pi K + 0}{n}\right)}$$

for $K = 0, 1 \& 2$ (ie, $K < n$)

$\therefore$ we have

$$R_0 = (4)^{1/2.5} e^{i\left(\frac{2\pi \times 0 + 0}{2.5}\right)}$$

$$= (4)^{1/2.5} e^{i0} = (4)^{1/2.5}$$

Using logarithms:

$$\frac{1}{2.5} \log(4) = 0.6021/2.5$$

$$= 0.2408$$

Using anti log:

$$\text{Antilog}\,(0.2408) = 1.741$$

$$\therefore \quad (4)^{1/2.5} = 1.741$$

(Verify result !)

Also,

$$R_1 = (4)^{1/2.5} \cdot e^{i\left(\frac{2\pi \times 1 + 0}{2.5}\right)}$$

$$= 1.741\, e^{i\,180^\circ/2.5} = 1.741\, e^{i144^\circ}$$

$$R_2 = (4)^{1/2.5} \cdot e^{i\left(\frac{2\pi \times 2 + 0}{2.5}\right)}$$

$$= 1.741 \cdot e^{i144^\circ \times 2}$$

$$= 1.741\, e^{i288^\circ}$$

- In the case of integer roots, for ex $\sqrt[3]{27}$, we had 3 roots for $K = 0, 1, 2$ and the roots repeated after that.

- Let us check whether the roots repeat, if we continue the process.

- We can calculate

$$R_3 = (4)^{1/2.5} e^{i\frac{2\pi \times 3 + 0}{2.5}} = 1.741\, e^{i72^\circ}$$

$$R_4 = (4)^{1/2.5} e^{i\frac{2\pi \times 4 + 0}{2.5}} = 1.741\, e^{i216^\circ}$$

$$R_5 = (4)^{1/2.5} e^{i\frac{2\pi \times 5 + 0}{2.5}} = 1.741\, e^{i0^\circ}$$

Hence $R_5 = R_0$ and the roots repeat after R_5 !

- Let us now verify our results.

$$(R_0)^{2.5} = (1.741)^{2.5} \doteq 4 \quad ✓$$

$$(R_1)^{2.5} = (1.741)^{2.5} \left(e^{i144^\circ}\right)^{2.5}$$

$$= 4 \cdot e^{i360^\circ} = 4 \quad ✓$$

$$(R_2)^{2.5} = (1.741)^{2.5} \left(e^{i288^\circ}\right) = 4 \quad ✓$$

$$(R_3)^{2.5} = (1.741)^{2.5} \left(e^{i72^\circ}\right)^{2.5}$$

$$= 4 \cdot e^{i180^\circ} = -4 \quad ✗$$

$$(R_4)^{2.5} = (1.741)^{2.5} \left(e^{i216^\circ}\right)^{2.5} = -4 \quad ✗$$

Note that mathematically,

$$\sqrt[2.5]{4} = (4)^{1/2.5} = (4)^{2/5} = \sqrt[5]{16}$$

R_0, R_1, R_2, R_3 & R_4 are valid roots of $\sqrt[5]{16}$; but only R_0, R_1 & R_2 are valid roots of $\sqrt[2.5]{4}$!!