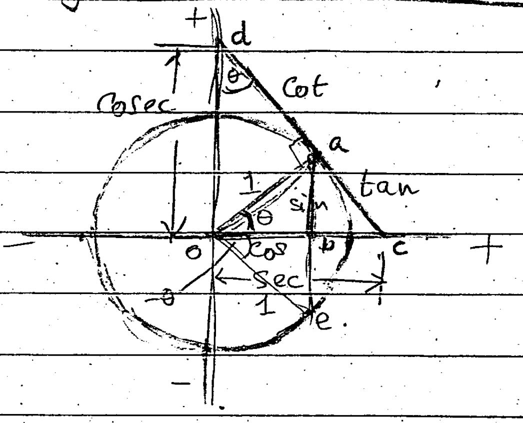
U3A Maths

24-Jun-2025 Term 2 / Week 9

Trigonometric Identities



By inspecting the above figure, we have,

$\Delta\, oab \Rightarrow \sin^2(\theta) + \cos^2(\theta) = 1$

or $\cos^2(\theta) = 1 - \sin^2(\theta)$

or $\sin^2(\theta) = 1 - \cos^2(\theta)$

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$\Delta\, oab$ & $\Delta\, oac$

$$\sec(\theta) = \frac{1}{\cos\theta}$$

Similarly,

$$\mathrm{cosec}(\theta) = \frac{1}{\sin(\theta)}; \quad \cot(\theta) = \frac{1}{\tan(\theta)}$$

$$\frac{\sin(\theta)}{\cos(\theta)} = \frac{OPP/HYP}{ADJ/HYP} = \frac{OPP}{ADJ} = \tan(\theta)$$

Also, $\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\cos(\theta)}{\sin(\theta)}$

$\sin(-\theta) = -\sin(\theta)$

$\cos(-\theta) = \cos(\theta)$

$\therefore \tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin(\theta)}{\cos\theta}$

$= -\tan(\theta)$

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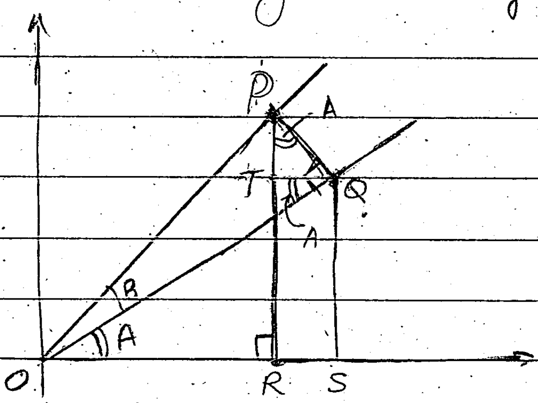
We have a very popular identity, namely

$\sin(A+B) = \sin(A)\cos(B) + \cos(A).\sin(B)$

Let us prove this geometrically



- The line PQ is drawn, so that it is perpendicular to OQ.
- Note the angle "A" identities.

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we have,

$$\sin(A+B) = \frac{PR}{OP} = \frac{PT + TR}{OP}$$

since $TR = QS$; $= \frac{PT + QS}{OP} = \frac{PT}{OP} + \frac{QS}{OP}$

$$= \frac{PT}{PQ} \cdot \underbrace{\frac{PQ}{OP}}_{\Delta\, OPQ} + \frac{QS}{OQ} \cdot \underbrace{\frac{OQ}{OP}}_{\Delta\, OPQ}$$

$= \cos(A).\sin(B) + \sin(A).\cos(B)$

Hence, we have,

$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$

Ex: Let $A = 30^\circ$ & $B = 60^\circ$

$\sin(A+B) = \sin(30^\circ + 60^\circ) = 1$

Also $\sin(30^\circ).\cos(60^\circ) + \cos(30^\circ)\sin(60^\circ)$

$= \frac{1}{2} \times \frac{1}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} = \frac{1}{4} + \frac{3}{4} = 1$

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Similarly, we can show that

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

Ex. 2- $A = 30^\circ \quad B = 60^\circ$

$\cos(30^\circ + 60^\circ) = \cos(90^\circ) = 0$

Also,

$\cos(30^\circ).\cos(60^\circ) - \sin(30^\circ)\sin(60^\circ)$

$= \sqrt{3}/2 \times 1/2 - 1/2 \times \sqrt{3}/2 = \underline{\underline{0}}$

Euler's Identity (theorem?)

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Euler's identity relates "trigonometry" to "algebra"!

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Let us now prove "$\sin(A+B)$" identity using Euler's Theorem

$$e^{i(A+B)} = \cos(A+B) + i\sin(A+B)$$

Also, $e^{i(A+B)} = e^{iA} . e^{iB}$

$= (\cos(A) + i\sin(A)).(\cos(B) + i\sin(B))$

$= \cos(A).\cos(B) + i\cos(A).\sin(B) + i\sin(A)\cos(B) + \underbrace{i^2}_{(-1)}\sin(A).\sin(B)$

$= [\cos(A).\cos(B) - \sin(A).\sin(B)] + i[\sin(A)\cos(B) + \cos(A)\sin(B)]$

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Equating real parts & imaginary parts respectively,

$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$

- We have proved both $\cos(A+B)$ & $\sin(A+B)$!
- We can also easily derive $\cos(A-B)$ & $\sin(A-B)$!!
- We can now derive some more identities.

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If $A = B$, then

$\sin(A+A) = \sin(2A)$

$= 2\sin(A).\cos(A)$

Also,

$\cos(A+A) = \cos(2A)$

$= \cos^2(A) - \sin^2(A)$

Expressed with 'θ' (instead of 'A')

$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$

$= (1 - \sin^2(\theta)) - \sin^2(\theta)$

$= 1 - 2\sin^2(\theta)$

$\therefore \sin^2(\theta) = \frac{1}{2}[1 - \cos(2\theta)]$

The above helps to convert "square" to normal (?) trigonometric function!! ex. Find $\underline{\sin^2(30^\circ)}$