U3A Maths

18-Nov-2025 Term 4 / Week 6

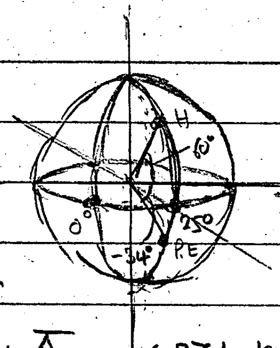
Spherical Law of Cosines

Review

Home work

Given that coordinates of Helsinki is $(\underset{N}{+60^\circ}, \underset{E}{+25^\circ})$ and Port Elizabeth is $(\underset{S}{-34^\circ}, \underset{E}{25^\circ})$ calculate the distance between the cities and the flight time assuming an average plane speed of 620 km/hr

Both cities are on the same longitude, ie, 25° E



∴ distance bet cities

$$d = (60^\circ - (-34^\circ)) \times \frac{\pi}{180} \times 6371 \text{ km}$$

$$= \underline{10{,}452 \text{ km}}$$

Flight time $= t = 10{,}452 \text{ km} / 620 \text{ km/hr}$

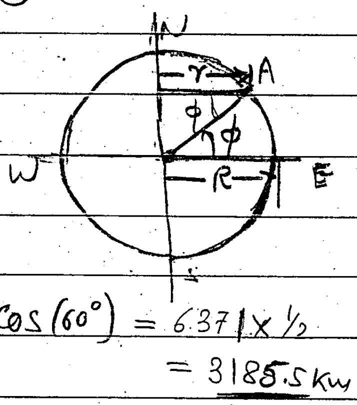
$$\left\{ \begin{matrix} s = d/t \\ \therefore t = d/s \end{matrix} \right\} \quad = 16.86 \text{ km}$$

(16 hrs 52 m)

- Hence, calculation of distances on the same longitude is straight forward.

- Similar calculations can be done for distances between two points on a given latitude.
- However, we should use the relevant radius of Earth at the given latitude.

- Earth radius at latitude 'ϕ'



$= r = R.\cos(\phi)$

ie, at $\phi = 60^\circ$

$r = 6371 \times \cos(60^\circ) = 6371 \times \frac{1}{2}$

$= 3185.5 \text{ km}$

Home work

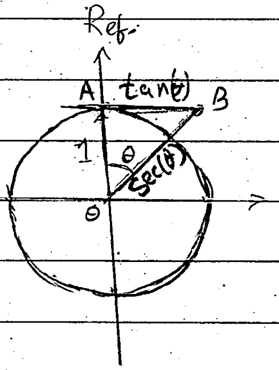
Calculate the flying distance between Sydney (34°S, 151°E) and CapeTown (34°S, 18°E)

∴ For calculating the distance between any two given points on earth, we use the "Spherical Law of Cosines"!

- We did use the spherical law of cosines, to calculate the solar hour angle (SHA). However, we did not go through the proof, since the modelling was complex! It required modelling of earth (sphere) declination of Earth, the sun and the local horizon!

- However, in the present case we are only dealing with the Earth!
- Let us first review the relevant (traditional) trigonometric concepts and equations in 2-dimensions!

- Let us consider an unit circle.



$\tan(\theta) = \frac{AB}{OA} = \frac{AB}{1}$

$\therefore AB = \tan(\theta)$

$\cos(\theta) = \frac{OA}{OB} = \frac{1}{OB}$

$\therefore OB = \frac{1}{\cos(\theta)} = \sec(\theta)$

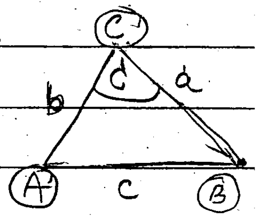
Also, using Pythagoras's theorem,

$$OA^2 + AB^2 = OB^2$$

i.e., $1^2 + \tan^2(\theta) = \sec^2(\theta)$

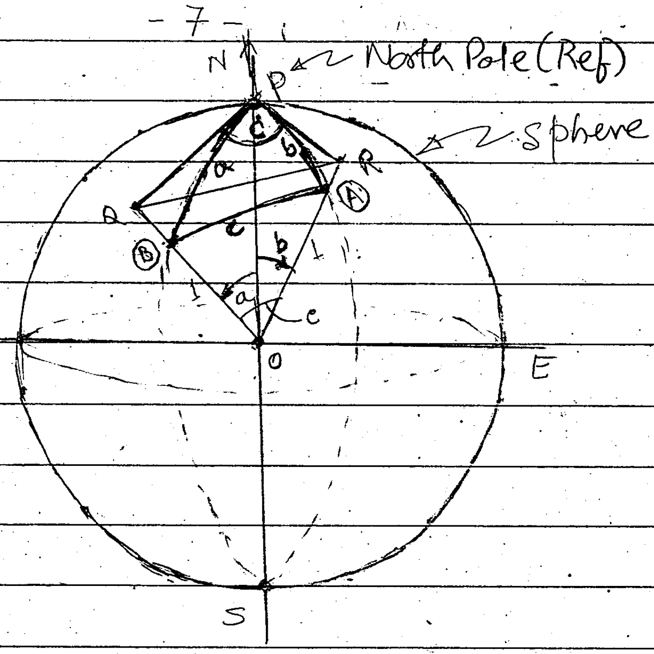
$$\therefore \boxed{\sec^2(\theta) - \tan^2(\theta) = 1}$$

We also have the Law of Cosines for any given triangle:



$$c^2 = a^2 + b^2 - 2ab\cos(d)$$

- We are now ready to derive the "<u>Spherical Law of Cosines</u>"



- We need to calculate the distance 'c' between 'A' and 'B'
- Let us construct a triangle using corresponding longitudes 'a' and 'b'
- Let "d" be the angle opposite to side 'c'
- PQ & OQ are tangent & secant of 'a'
PR & OR are tangent & secant of 'b'

- Note that lines (arcs!) 'a' and 'b' also corresspond to spherical angles 'a' & 'b', with sphere centre (O) & Pole (P) as references.
- OA & OB are <u>unit</u> Radii, since 'A' & 'B' are on sphere surface.
- Considering triangles OPQ & OPR,

$PQ = \tan(a)$ & $OQ = \sec(a)$

$PR = \tan(b)$ & $OR = \sec(b)$

- Using law of cosines for the (plane) triangle PQR,

$$QR^2 = PQ^2 + PR^2 - 2(PQ)(PR)\cos(d)$$

- Spherical angle 'd' is also the angle between tangents.

-9-

Substituting for 'PQ' & 'PR',

$$QR^2 = \tan^2(a) + \tan^2(b) - 2\tan(a)\tan(b)\cos(d)$$

$\Rightarrow$ Eqn (1)

• Using law of cosines for 'plane' triangle 'OQR',

$$QR^2 = OQ^2 + OR^2 - 2(OQ)(OR)\cos(c)$$

Note the 'c' is radial angle corresspording to distance A to B on the sphere.

-10-

Substituting for 'OQ' & 'OR'

$$QR^2 = \sec^2(a) + \sec^2(b) - 2\sec(a)\sec(b)\cos(c)$$

$\Rightarrow$ Eqn. (2)

We can eliminate $(QR)^2$ by equating Eqn(1) & Eqn(2)

∴

$$\tan^2(a) + \tan^2(b) - 2\tan(a)\tan(b)\cos(d) = \sec^2(a) + \sec^2(b) - 2\sec(a)\sec(b)\cos(c)$$

Collecting the terms, we can write

-11-

$$[\sec^2(a) - \tan^2(a)] + [\sec^2(b) - \tan^2(b)] + 2\tan(a)\tan(b)\cos(d) - 2\sec(a)\sec(b)\cos(c) = 0$$

We have,

$$\sec^2(a) - \tan^2(a) = 1 \quad \& \quad \sec^2(b) - \tan^2(b) = 1$$

∴

$$\overset{1}{\cancel{2}} + \overset{1}{\cancel{2}}[\tan(a)\tan(b)\cos(d)] - \underset{1}{\cancel{2}}[\sec(a)\sec(b)\cos(c)] = 0$$

∴

$$1 + \frac{\sin(a)}{\cos(a)}\,\frac{\sin(b)}{\cos(b)}\cdot\cos(d) - \frac{1}{\cos(a)}\cdot\frac{1}{\cos(b)}\cdot\cos(c) = 0$$

-12-

Multiplying by $[\cos(a).\cos(b)]$

$$\cos(a)\cdot\cos(b) + \sin(a)\sin(b)\cos(d) - 1\cdot\cos(c) = 0$$

∴ Finally, we have the Spherical Law of Cosines!

$$\boxed{\cos(c) = \cos(a)\cos(b) + \sin(a)\sin(b)\cos(d)}$$

Spherical Law of Cosines corressponds to the spherical triangle ABC!