U3A Maths

31-Mar-2026 Term 1/Week 10

Cube root & Higher order Roots

- We calculated the square root using the Babylonian Method as below:

$x = \sqrt{S}$ where S is given

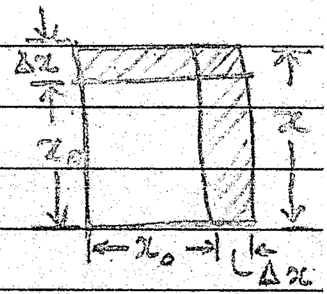
Let x_0 be the initial guess. (Using the "table of Squares")

The new estimate is

$$x_1 = x_0 + \frac{S - x_0^2}{2x_0}$$

- The above equation gives surprisingly good results
- of course, the "iterations" can be continued for higher precision.

- We derived the above equation (Algorithm) using geometric / algebraic method



We have,

Total Area = S

we also have,

Total Area

$= x_0^2 + 2x_0 \cdot \Delta x + (\Delta x)^2$

Neglecting $(\Delta x)^2$ term since Δx is small (!), we have

$$S \cong x_0^2 + 2x_0 \Delta x$$

$$\therefore \Delta x = \frac{S - x_0^2}{2x_0}$$

$\therefore$ The new estimate is

$$x_1 = x_0 + \Delta x$$

$$\boxed{x_1 = x_0 + \frac{S - x_0^2}{2x_0}}$$

We can re-write the above equation in an alternative form - which is computationally more efficient

$$x_1 = \frac{1}{2}\left[x_0 + \frac{S}{x_0}\right]$$

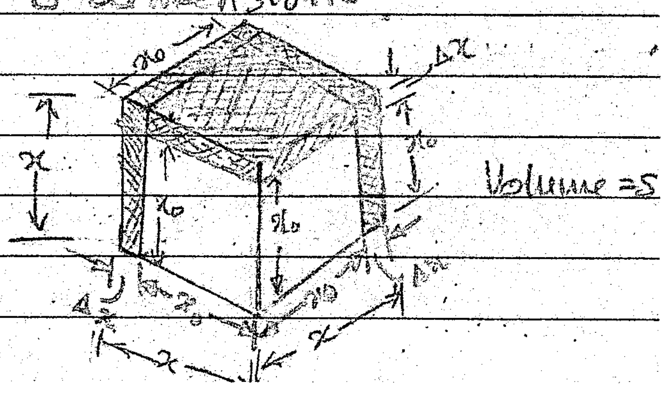
[Proof is left to the reader!]

Cube Root

- We can extend the Babylonian algorithm for cube root!

Let $x = \sqrt[3]{S}$

- Let x_0 be the initial estimate
- We can derive the algorithm for cube root, by extending the Babylonian method to 3 dimensions!



We have,

Total volume $= S$

we also have

Total volume,

$= x_0^3 + 3x_0^2\Delta x + 3x_0(\Delta x)^2 + (\Delta x)^3$

Neglecting $(\Delta x)^2$ and $(\Delta x)^3$ terms

$$S \simeq x_0^3 + 3x_0^2 \cdot \Delta x$$

$$\therefore \Delta x = \frac{S - x_0^3}{3x_0^2}$$

Hence, the new estimate is

$$x_1 = x_0 + \Delta x$$

$$= x_0 + \frac{S - x_0^3}{3x_0^2}$$

ie, $$\boxed{x_1 = x_0 + \frac{(S - x_0^3)}{3x_0^2}}$$

The alternative form is

$$\boxed{x_1 = \frac{1}{3}\left[2x_0 + \frac{S}{x_0^2}\right]}$$

Ex.1 Find $\sqrt[3]{10}$

we have $S = 10$

$\therefore$ Let initial Estimate be

$x_0 = 2$

Cube Table
$1^3 = 1$
$2^3 = 8$
$3^3 = 27$

$$\therefore x_1 = \frac{1}{3}\left[2 \times 2 + \frac{10}{(2^2)}\right]$$

$$= \frac{1}{3}\left[4 + \frac{10}{4}\right] = \frac{1}{3}\left[4 + 2.5\right] \cong 2.16$$

Using calculator $\sqrt[3]{10} = 2.154$

For given value of 'S':

Square Root : $x_1 = \frac{1}{2}\left[x_0 + \frac{S}{x_0}\right]$

Cube Root : $x_1 = \frac{1}{3}\left[2x_0 + \frac{S}{x_0^2}\right]$

We can now logically extend the above algorithm for higher roots!

Fourth Root : $x_1 = \frac{1}{4}\left[3x_0 + \frac{S}{x_0^3}\right]$

Fifth Root : $x_1 = \frac{1}{5}\left[4x_0 + \frac{S}{x_0^4}\right]$

nth Root :

$$x_1 = \frac{1}{n}\left[(n-1)x_0 + \frac{S}{x_0^{(n-1)}}\right]$$

Ex.2

Calculate $\sqrt[5]{48}$

We have

$S = 48$

Initial Estimate $x_0 = 2$

$1^5 = 1$

(48) $2^5 = 32$

$3^5 = 243$

$$x_1 = \frac{1}{5}\left[4 \times 2 + \frac{48}{24}\right] = \frac{1}{5}\left[8 + 3\right]$$

$$= 2.2$$

Using calculator: $\sqrt[5]{48} = 2.1689$

Home work

(1) Perform on more iteration for Ex.(2), ie, $\sqrt[5]{48}$

(2) Calculate $\sqrt[4]{18}$ or $(18)^{1/4}$

(3) Calculate $\sqrt[3]{128}$

Note: $\sqrt[3]{128} = \sqrt[3]{64 \times 2} = \sqrt[3]{64} \times \sqrt[3]{2} = 4 \times \sqrt[3]{2}$