U3A Maths

16-Jun-2026 Term 2 / Week 9

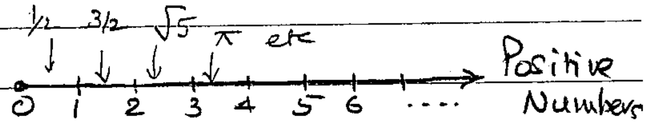
Complex Numbers
(An Alternative View)

- The "Alternative View" demonstrates that a Complex Number is a logical extension of the number system in every day use!

- Let us consider a set of positive numbers, such as,
  $1, 2, 3, \ldots ; \frac{1}{2}, \frac{1}{5}, \frac{3}{8} \ldots ; \sqrt{2}, \sqrt{5} \ldots ;$
  (whole numbers) (Fractions) (Irrational numbers)

  and also π, e, ... (Transcedental Numbers)
  3.14159.. 2.718...

- These numbers can be illustrated in graphical form as below:



- All positive numbers can be plotted on the above line (axis)

- A set is said to be "closed" for a given arithmetic operation, provided the resulting number is also a part of the set.

Add: $1+2=3$ etc (closed)
Mult: $3\times 6 = 18$ etc (closed)
Divide: $3 \div 2 = 1.5$ etc (closed)

Square Root $\sqrt{2} = 1.414..$ (closed)
Subtract: $3-2=1$ (closed)
$4-6=-2$ (Not closed)

- The set of positive numbers is not necessarily closed for subtraction.

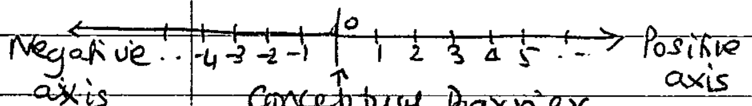
- We will need to extend our set to include negative numbers
- Negative numbers are not easy to comprehend in practice, since negative numbers do not exist in the "Real" world!

- In the western world, negative numbers were first documented by Cardano (Italian, 1545 AD)
- However, even the famous mathematicians, such as, Des Cartes (1596-1650 AD) and Leibniz (1646-1716 AD) were not comfortable with them. The final result had to positive - Always!

- In fact, the term "Negative" itself has deragatory connotation!
- Cardano called them "fictitious" numbers!
- Michael Stifel (1544 AD) called them "Absurd" numbers!

- Johannes Scheubel (1551 AD) was one of the first Europeans to use the term "Negative"!
- Finally, Brahmagupta (Indian Mathematician, 628 AD) used the terms "Fortune" or "Asset" for positive numbers and "Debt" for Negative numbers. So, Indians had no problems with Negative numbers!

- Let us call the new set as the "Set of Real Numbers"



- The "Set of Real numbers" is not closed for square root of negative numbers!

- For Ex: $\sqrt{-2} = ?$

This value cannot be shown (plotted) graphically?!
- Geometry cannot accomodate "square root of negative numbers.

- Hence, Des Cartes (father of Geometry) used the term "Imaginary" numbers. Hence, we are stuck with this term and eternal confusion!

- Square root of negative numbers can be formalised by denoting (defining?) the symbol $i = \sqrt{-1}$

- Then we have, for Ex.

$$\sqrt{-4} = \sqrt{(-1)(4)} = \sqrt{-1}\sqrt{4} = i4 \text{ or } 4i$$

The above convention is useful while working with roots of negative numbers.

- The above convention was used by Bombelli (collegue of Cardano) when solving for

(real) root of cubic equations.

- Bombelli was solving for the real root of the cubic equation below.

$$x^3 = 15x + 4$$

Using his method for calculating the "real" root (there is at least one, for cubic Eqns), he obtained the result

$$x = \sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}}$$

- He knew that the answer is a real no.
- He solved the above using

$$\sqrt[3]{2+\sqrt{-121}} = a+bi$$
$$\sqrt[3]{2-\sqrt{-121}} = a-bi$$

- He calculated 'a' & 'b' using

$$2+\sqrt{-121} = (a+bi)^3$$
$$2-\sqrt{-121} = (a-bi)^3$$

and found that $a=2$ & $b=1$

$\therefore$ The answer (the root) is

$$x = (a+\cancel{bi}) + (a-\cancel{bi})$$
$$= 2a = \underline{\underline{4}}$$

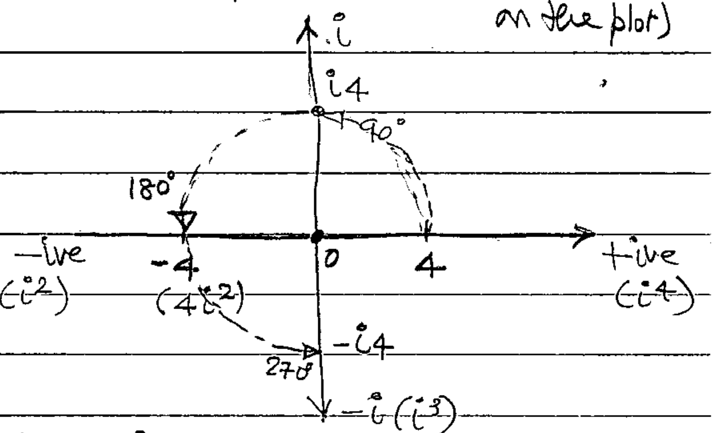
- While solving, he used the following identities to simplify the expressions

$$i = \sqrt{-1}$$
$$\therefore\ i^2 = -1 \quad \text{and} \quad i^3 = i^2 \cdot i = -i$$

Also, $i^4 = +1$

- The above identities provide a way to graphically plot square root of negative numbers!

- $4 \times -1 = -4$ or $4i^2$ (as shown on the plot)



(-1) or i^2 shifts the value by 180°

- Hence, "i4" or $(\sqrt{-1})\,4$ shifts the value by 90°!

- Similarly $(i^3)\,4$ or $-i4$ shifts the value by 270°!

- Note that $i4 \Longleftrightarrow (\sqrt{-1})\,4$ ← Identical to

$$i^2 4 \Longleftrightarrow (\sqrt{-1})^2 4 = -1$$

etc

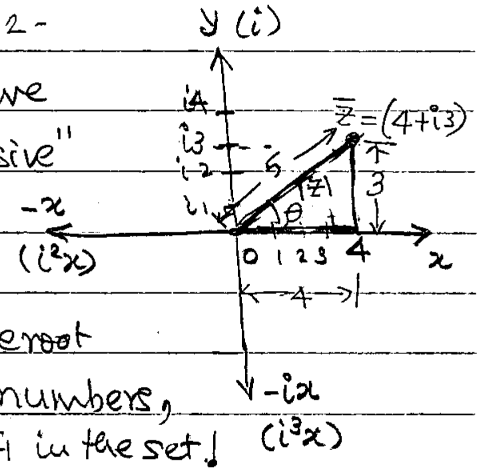
- We can now plot the value, say $4+\sqrt{-9}$

$$4+\sqrt{-9} = 4 + (\sqrt{-1})\times 3$$
$$= (4+i3)$$

- The comprehensive number is also called the "Complex" number

$\bar{Z} = x + iy$ < $x \Rightarrow$ Real Part! ; $iy \Rightarrow$ Imaginary Part!

- We now have a "comprehensive" number set which includes square root of negative numbers, by including $\sqrt{-1}$ in the set!



- Let us say the comprehensive number in our example is

$$\bar{Z} = (4+i3) \text{ or } (x+iy)$$

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- Another way of representing the comprehensive number is

$\bar{Z} = |Z|\,\angle\theta$ (Magnitude, Angle) where $|Z| = \sqrt{x^2+y^2} = \sqrt{4^2+3^2} = 5$

$\theta = \tan^{-1}(y/x) = \tan^{-1}\left(\frac{3}{4}\right) = 36.8°$