U3A Maths

11-Aug-2026 Term 3 / Week 4

Complex Plots - Conformal Mapping

- The "3-D plots" and "Domain coloring" provided for impressive, colourful and "beautiful" plots.
- However, even simple plots are also very useful for Engineering design.
- One such plot is to plot Roots (zeros) and Poles (infinity) on Real - Imaginary axes

Ex. Plot the roots and poles of the following function.

$$f(z) = \frac{(z-1)}{(z^2+z+1)}$$

$f(z) = 0$ when $(z-1) = 0$

The root is $z_0 = 1$ (say)

$f(z) = \infty$ when denominator is 0

ie, $z^2 + z + 1 = 0$

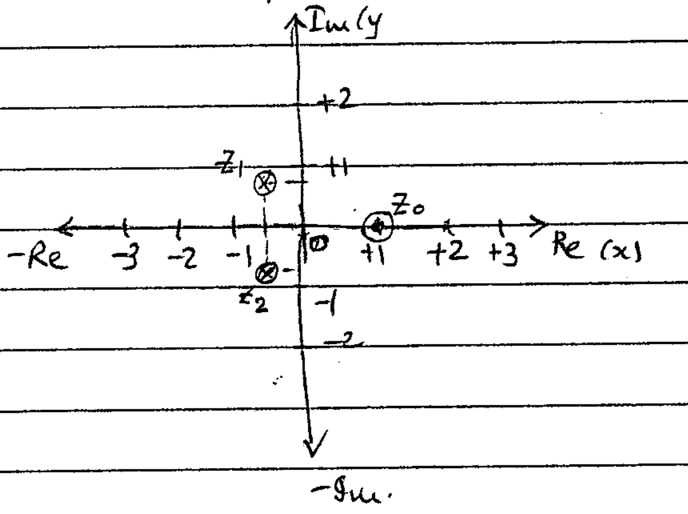
$$z = \frac{(-1) \pm \sqrt{(1)^2 - 4\times1\times1}}{2\times1}$$

Poles are

$z_1 = (-0.5 + i\,0.866)$ &

$z_2 = (-0.5 - i\,0.866)$

- We can plot the results on a simple x-y plot



- The control system design engineers use the above plot to determine the stability of control systems. For Ex, temperature control, speed control etc.

- The location of "poles" determine the stability of the control system.
- It is often called "Root Locus" plots, since Poles correspond to roots of the denominator!
- In simple terms:
  - "Negative real part" provides for a stable system.
  - "Positive real part" results in an unstable system.
  - "Positive / Negative imaginary part" results in an oscillatory system!
- More analysis can be done using "Root Locus".

Conformal Mapping

- Certain complex functions have a special use in engineering design and analysis

- One such very famous function is called "Joukowski Function" or "Joukowski Transformation" !

- Nikolai Joukowski (Zhukovsky) was a Russian scientist (17-Jan-1847 to 17-Mar-1921), who did

pioneering work in hydrodynamic & aerodynamics. He is best known for "Aerofoil" design used in planes, helicopters, turbines etc

- He is also known as the "Father of Russian Aviation".
- He provided a mathematical expression for the lift provided by a aerofoil.

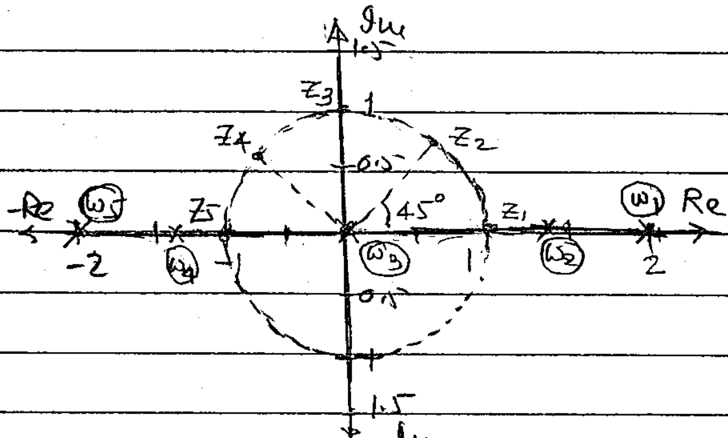
- An aerofoil design and analysis is done using Joukowski's Equation Conformal Mapping

- Joukowski function is

$$\omega = f(z) = z + 1/z$$

- Unlike domain colouring, we will plot the function only for the points on a unit circle. (There is a mathematical justification for this, but for the present we will take a rain check on that!)

- Let us calculate the function values for points on the unit circle



For Ex,

for $z = z_1 = 1\angle 0°$ $\quad \omega = \omega_1 = 1\angle 0° + 1/1\angle 0°$

$$= 1 + 1 = 2$$

$z = z_2 = 1\angle 45°$,

$$\omega = \omega_2 = 1\angle 45° + 1/1\angle 45°$$

using Euler's form

$$= 1e^{i45°} + \frac{1}{1.e^{i45°}}$$

$$= 1.e^{i45} + 1.e^{-i45°}$$

$$= (\cos 45° + i\sin 45°) + (\cos 45° - i\sin 45°)$$

$$= 2\cos 45° = 1.414$$

The function values are as listed below:

	z	$\omega = f(z)$	
(z_1)	$1\angle 0^\circ$	$2\angle 0^\circ$	(ω_1)
(z_2)	$1\angle 45^\circ$	$1.414\angle 0^\circ$	(ω_2)
(z_3)	$1\angle 90^\circ$	0	(ω_3)
(z_4)	$1\angle 135^\circ$	$-1.414\angle 0^\circ$	(ω_4)
(z_5)	$1\angle 180^\circ$	-2 or $2\angle 180^\circ$	(ω_5)

- Hence, the function values are a straight line from +2 to −2 (on 'Re' axis)

- The plot of 'ω' values are shown in Fig (on Page 8)

- The plot of 'ω' values is called the "Conformal Map".

- The question is, what is the use of such a plot?!

- To illustrate its use, let us create a new plot for the points on a circle, whose centre is offset by $\Delta x = 0.04$ & $\Delta y = 0.4$

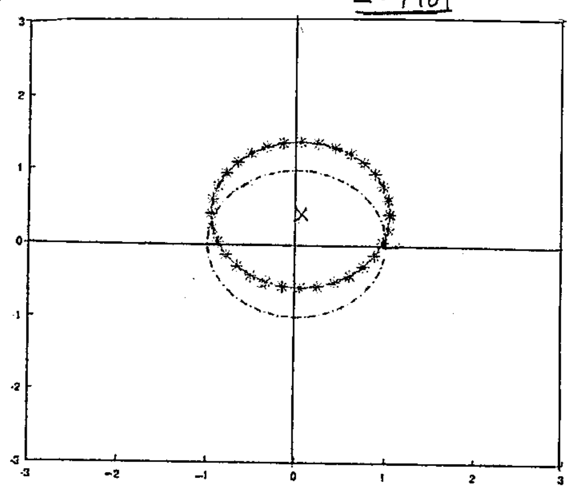
- That is, for $z_1 = (1 + i0)$,

$$z_1 (\text{offset}) = (1 + 0.04) + i(0 + 0.4) = (1.04 + i0.4)$$

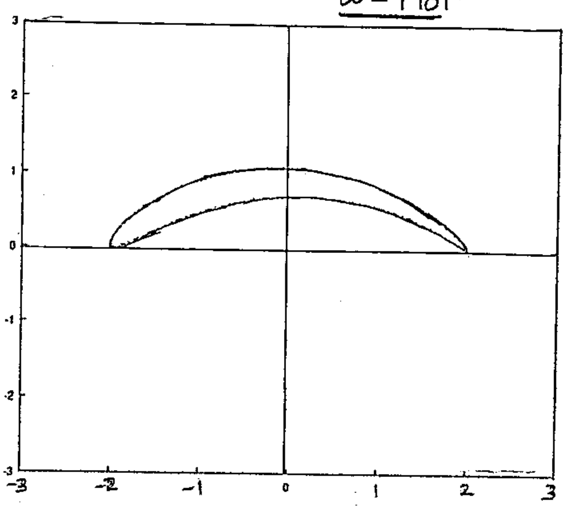
{ Home Work! }

- The results are shown below!

Z-Plot



ω-Plot



- The ω-plot shows an aerofoil! The shape of the aerofoil can be changed by changing the offset of the centre.

- Infact, the velocity & pressure of the air flow on the z-plot can also be transformed to the corresponding values in the ω-plot to establish the "lift" developed by the aerofoil.

- For an interactive demo, visit web site "https://complex-analysis.com/content/joukowsky_airfoil.html"