U3A Maths

25-Aug-2026 Term 3 / week 6

Joukowski Airfoil

Joukowski Transform

- $w = f(z) = z + \frac{1}{z}$
  where $z = (x+iy)$ & $w = (u+iv)$
- Let us have a look at the properties of the transform
- Let us look at $f(z)$ values for the following z-values. For simplicity, let us assume 'Real' values for z. This is adequate for our objective.

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z	$f(z) = z + 1/z$	% Change
1	$1 + \frac{1}{1} = 2$	100%
2	$2 + 1/2 = 2.5$	25%
3	$3 + 1/3 = 3.333$	11.1%
4	$4 + 1/4 = 4.25$	6.25%
5	$5 + 1/5 = 5.2$	4%
6	$6 + 1/6 = 6.167$	2.8%
7	$7 + 1/7 = 7.143$	2%
10	$10 + 1/10 = 10.1$	1%
15	$15 + 1/15 = 15.067$	0.45%

- Note that $f(z) \simeq z$ for $|z| > 10$. In other words, Zoukowski transformation has no effect on values away from the airfoil!

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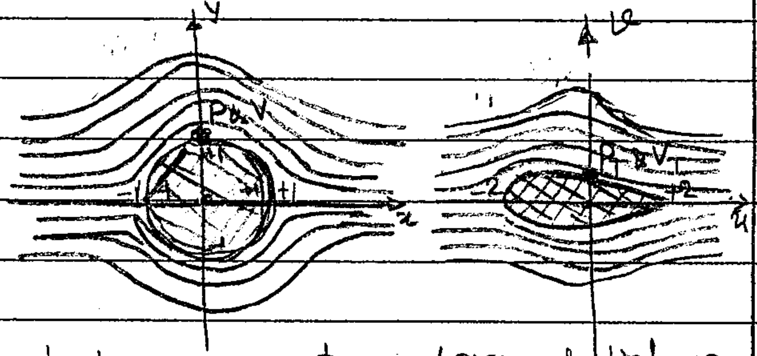
- Another feature of the Zoukowski transform is that, the other flow parameters such as pressure and velocity can also be transformed. The transformed values corresponds to the actual physical values around the air foil!



- Let us say transformed values are $P \Rightarrow P_T$ & $V \Rightarrow V_T$
- In practice P & V are complex, as they have magnitude and direction.

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- The values of P_T & V_T correspond to the actual pressure & velocity at the corresponding point on the air foil!
- Also, transformed values of pressure & velocity at points sufficiently away from the air foil will be same as the values for the flow around the cylinder!
- Incidentally, it can be shown that the Zoukowski's transformation is not valid for $|z| \leq 1$, that is, inside the cylinder & air foil!

- The above restriction does not affect our model, as we are interested only in the values outside the cylinder & air foil!

- The detailed equations for calculating the parameters around the air foil are beyond the scope of this presentation.

- It is sufficient to say that, it involves solving "second order partial differential equations with boundary conditions" in complex form.

  Navier-Stokes Equation —

$$\rho\left(\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V}\right) = -\nabla P + \mu\nabla^2\vec{V} + \rho g$$

- The main objective of the air foil equations is to establish the "lift" force exerted on an air foil or wing of a plane.

- A simplified version of the equation is popularly used for Lift calculations.

Equation for Lift (force)

- The fundamental equation for fluid flow was derived based on Bernoulli's Principle.

  Daniel Bernoulli (1700 – 1782)

  Swiss Mathematician

[Note: Newton (1643-1727)]

- The Bernoulli's principle (1738 AD) states that the fluid pressure decreases as the flow speed increases.

- However, the popular Bernoulli's equation was developed by Euler! (in 1752 AD)

- Bernoulli's Equation :

$$\underbrace{P}_{\text{Fluid Pressure Energy}} + \underbrace{\tfrac{1}{2}\rho V^2}_{\text{Kinetic Energy}} + \underbrace{\rho g h}_{\text{Potential Energy}} = \text{Constant}$$

where, P - Fluid Pressure

ρ - Fluid density

V - Fluid Velocity

g - Accln due to gravity

h - height or elevation of fluid.

- For Air foil applications, the elevation (h) can be considered as a constant, hence we have

$$P + \tfrac{1}{2}\rho V^2 = \text{Constant.}$$

- Considering upper & bottom surface of air foil

$$P_u + \tfrac{1}{2}\rho V_u^2 = P_l + \tfrac{1}{2}\rho V_l^2 = \text{Const}$$

$$\therefore \underset{(\text{Force})}{\text{Lift}} = (P_l - P_u) = \tfrac{1}{2}\rho\left(V_u^2 - V_l^2\right)$$

- Hence, to obtain "Lift", the velocity of the fluid on the top surface must be higher than the velocity of the fluid at the bottom surface.

(This is what Air foil achieves!)

- A simplified equation for Lift is as below

$$F_L = C_L \cdot (\tfrac{1}{2}\rho V^2) \cdot S$$

where,

F_L - Lift Force

C_L - Lift coefficient

ρ - Fluid density

V - Velocity of fluid (Velocity of the Plane!)

S - surface area of the wing

- The key factor in the above equation is C_L. This includes the effects of
  - shape of the air foil
  - difference in velocities ($V_2 - V_1$), which is modelled as "Circulation"!
  - Angle of attack or incline angle of the air foil

- For a typical air foil, C_L values are as below:

Angle of attack ⇒	0°	5°	10°	18°	>19°
C_L ⇒	0.2	0.5	0.95	1.5	(Stall)

(for F 117A fighter plane) C_L decreases!

Ex.

Calculate the lift generated by a plane for the following data.

- Air density (ρ) = 1.225 kg/m³
- Air velocity (V) = 50 m/s (180 km/h)
- Wing Area (S) = 20 m²
- Lift coefficient (C_L) = 0.5 (Approx. 5° angle of attack)

Using Lift Equation

$$F_L = 0.5 \times \frac{1}{2} \times 1.225 \times (50)^2 \times 20$$

$$= 15{,}312.5 \text{ Newtons (N)}$$

or ($\simeq 15.3$ kN)

Homework

An aircraft wing has a chord length (that is, circumference of air foil) of 10 m and must generate a lift force of 3560 kN at a take off speed of 282 km/h. Given that Lift coefficient (C_L) is 1.0 and ρ = 1.225 kg/m³, find the wing span (length).

(Ans: 94.72 m)