17-Nov-26 Term 4 / Week 6

Matrix Inversion
(Cofactor Matrix Method)

- We used the 'ready made' calculator for matrix inversion to solve simultaneous equations
- Let us now explore the concepts in matrix inversion and the methods of computation.
- For single variables, the inversion is defined as below:
  For a 2,
  ∴ Inverse of 'a': $a^{-1} = \frac{1}{a}$ ie, $(2)^{-1}$ or $\frac{1}{2}$

Also, we have,

$a \times a^{-1} = 2 \times \frac{1}{2} = 1.$

Hence, any variable multiplied by its inverse is always equal to 1.

- In the case of a matrix,

$$[A]\,[A]^{-1} = [I]$$

where [I] is called an "Identity Matrix". It has all diagonal values as '1's and other values as '0's.

- Let us consider a (2×2) matrix

$$[A] = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

let $[A]^{-1}$ be $= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

- By definition, we have

$$[A]\,[A]^{-1} = [I]$$

ie,

$$\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Using matrix multiplication,

$$\begin{bmatrix} (1\times a + 2\times c) & (1\times b + 2\times d) \\ (-3\times a + 4\times c) & (-3\times b + 4\times d) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By equating the corresponding matrix elements, we have,

$(1\times a + 2\times c) = 1$ & $(-3\times a + 4\times c) = 0$

ie,

$a + 2c = 1$

$-3a + 4c = 0$ or $c = 3a/4$

Substituting,

$a + 2(3a/4) = 1$

$a[1 + 3/2] = 1$

$\therefore a = \frac{1}{(5/2)} = \boxed{\frac{2}{5}}$

Also $c = \frac{3a}{4} = \frac{3}{4} \times \left(\frac{2}{5}\right) = \boxed{\frac{3}{10}}$

Also, we have

$1\times b + 2\times d = 0$

$-3\times b + 4\times d = 1$

Solving, we get

$b = \boxed{-1/5}$ & $d = \boxed{1/10}$

Hence, the inverted matrix is

$$[A]^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2/5 & -1/5 \\ 3/10 & 1/10 \end{bmatrix}$$

We can also write $[A]^{-1}$ as,

$$[A]^{-1} = \begin{bmatrix} 4/10 & -2/10 \\ 3/10 & 1/10 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}$$

- Let us now check the results.

$$[A][A]^{-1} = [I]$$

$$\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} \times \frac{1}{10} = \frac{1}{10}\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- The above method gets unwieldy for higher order matrices. for Eg, for a (3×3) matrix, we need to solve for 3 sets of simultaneous equations in 3 variables! Hence we need a better and more systematic method!

Matrix inversion using the Cofactor matrix

- In cryptic form, we can write the method as below.
For given matrix [A],

$$[A]^{-1} = \frac{1}{|A|}[C]^T$$

where,

|A| - is called the "determinant" of [A]

Determinant is a single number representative of a square matrix.

Notes

- Determinant and hence the inverse can only be calculated for square matrices, for Ex (2×2), (3×3), (4×4) etc

- Inverse of a matrix is undefined, if the determinant is zero, as it results in division by zero!

- [C] is called the "cofactor" matrix and is established as below.

- For each element (say i^{th} row & j^{th} column) of the given matrix [A],

- the corresponding element (ie, i^{th} row & j^{th} column) of the "Cofactor matrix" is the determinant of the "Minor matrix" multiplied by $(-1)^{(i+j)}$

- Minor matrix is obtained by ignoring (deleting) the i^{th} row & j^{th} column of the given ([A]) matrix.

• The symbol 'T', indicates "transpose" of a matrix, i.e., rows & columns are interchanged.

For Ex:

$$[A] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad [A]^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Determinant of a (2×2) matrix

$$[A] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$|A| = (a \times d) - (c \times b)$$

Note

Downward arrow is positive & Upward arrow is negative

Ex.1 Find the determinants

(a) $$[A] = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$$

$$|A| = (1 \times 4) - (3 \times 2)$$
$$= 4 - 6 = -2$$

(b) $$[A] = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$$

$$|A| = (1 \times 4) - (-3 \times 2)$$
$$= 4 + 6 = 10$$

(c) $$[A] = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$$

$$|A| = (1 \times (-4)) - (3 \times 2) = -10$$

Determinant of a 3×3 matrix

$$[A] = \begin{bmatrix} a^{+} & b^{-} & c^{+} \\ d & e & f \\ g & h & k \end{bmatrix}$$

$$|A| = a \times \begin{vmatrix} e & f \\ h & k \end{vmatrix} - b \times \begin{vmatrix} d & f \\ g & k \end{vmatrix} + c \times \begin{vmatrix} d & e \\ g & k \end{vmatrix}$$

$$= a \cdot (ek - hf) - b(dk - gf) + c(dk - eg)$$

Row & col of 'a' ignored; Row & Col of 'b' ignored; Row & col of 'c' ignored

Ex.2 find determinants of

(a) $$[A] = \begin{bmatrix} 1^{+} & 2^{-} & 3^{+} \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$|A| = 1 \times \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \times \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \times \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1(5 \times 9 - 8 \times 6) - 2(4 \times 9 - 7 \times 6) + 3(4 \times 8 - 7 \times 5)$$
$$= (45 - 48) - 2(36 - 42) + 3(32 - 35)$$
$$= -3 + 12 + 3(-3) = 0$$!!

(b) $$[A] = \begin{bmatrix} 6 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{bmatrix}$$

Home Work! (Ans: -306)

Notes: Even though cofactor method for inversion sounds quite verbose and complicated, mainly due to the introduction new terms/concepts, it is fairly straight forward after some familiarity & examples.